MATH 311

Topics in Applied Mathematics I

**Lecture 17:
Nullity of a matrix.
Basis and coordinates.
Change of basis.**

Rank of a matrix

Definition. The **row space** of an $m \times n$ matrix A is the subspace of $\mathbb{R}^n$ spanned by rows of A. The **column space** of A is a subspace of $\mathbb{R}^m$ spanned by columns of A.

The row space and the column space of A have the same dimension, which is called the **rank** of A.

Theorem 1 Elementary row operations do not change the row space of a matrix.

Theorem 2 If a matrix A is in row echelon form, then the nonzero rows of A form a basis for the row space.

Theorem 3 The rank of a matrix is equal to the number of nonzero rows in its row echelon form.

Nullspace of a matrix

Let $A = (a_{ij})$ be an $m \times n$ matrix.

Definition. The **nullspace** of the matrix A, denoted $N(A)$, is the set of all n-dimensional column vectors $\mathbf{x}$ such that $\boxed{A\mathbf{x} = \mathbf{0}}$.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The nullspace $N(A)$ is the solution set of a system of linear homogeneous equations (with A as the coefficient matrix).

Theorem $N(A)$ is a subspace of the vector space $\mathbb{R}^n$.

Definition. The dimension of the nullspace $N(A)$ is called the **nullity** of the matrix A.

rank + nullity

Theorem The rank of a matrix A plus the nullity of A equals the number of columns in A.

Sketch of the proof: The rank of A equals the number of nonzero rows in the row echelon form, which equals the number of leading entries.

The nullity of A equals the number of free variables in the corresponding homogeneous system, which equals the number of columns without leading entries in the row echelon form.

Consequently, rank+nullity is the number of all columns in the matrix A.

Problem. Find the nullity of the matrix
$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \end{pmatrix}.$$

Clearly, the rows of A are linearly independent. Therefore the rank of A is 2. Since
$$(\text{rank of } A) + (\text{nullity of } A) = 4,$$
it follows that the nullity of A is 2.

Basis and dimension

Definition. Let V be a vector space. A linearly independent spanning set for V is called a **basis**.

Theorem Any vector space V has a basis. If V has a finite basis, then all bases for V are finite and have the same number of elements (called the *dimension* of V).

Example. Vectors $\mathbf{e}_1 = (1, 0, 0, \ldots, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0, \ldots, 0, 0), \ldots, \mathbf{e}_n = (0, 0, 0, \ldots, 0, 1)$ form a basis for $\mathbb{R}^n$ (called *standard*) since

$$(x_1, x_2, \ldots, x_n) = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + \cdots + x_n \mathbf{e}_n.$$

Basis and coordinates

If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then any vector $\mathbf{v} \in V$ has a unique representation

$$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n,$$

where $x_i \in \mathbb{R}$. The coefficients $x_1, x_2, \ldots, x_n$ are called the **coordinates** of $\mathbf{v}$ with respect to the ordered basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

The mapping

vector $\mathbf{v} \mapsto$ *its coordinates* $(x_1, x_2, \ldots, x_n)$

is a one-to-one correspondence between V and $\mathbb{R}^n$. This correspondence respects linear operations in V and in $\mathbb{R}^n$.

Examples. • Coordinates of a vector
$\mathbf{v} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ relative to the standard
basis $\mathbf{e}_1 = (1, 0, \ldots, 0, 0)$, $\mathbf{e}_2 = (0, 1, \ldots, 0, 0)$,...,
$\mathbf{e}_n = (0, 0, \ldots, 0, 1)$ are $(x_1, x_2, \ldots, x_n)$.

• Coordinates of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_{2,2}(\mathbb{R})$
relative to the basis $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$,
$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are (a, c, b, d).

• Coordinates of a polynomial
$p(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} \in \mathcal{P}_n$ relative to
the basis $1, x, x^2, \ldots, x^{n-1}$ are $(a_0, a_1, \ldots, a_{n-1})$.

Weird vector space

Consider the set $V = \mathbb{R}_+$ of positive numbers with a nonstandard addition and scalar multiplication:

$$\boxed{x \oplus y = xy} \quad \text{for any } x, y \in \mathbb{R}_+.$$
$$\boxed{r \odot x = x^r} \quad \text{for any } x \in \mathbb{R}_+ \text{ and } r \in \mathbb{R}.$$

This is an example of a vector space.

The zero vector in V is the number 1. To build a basis for V, we can begin with any number $v \in V$ different from 1. Let's take $v = 2$. The span $\mathrm{Span}(2)$ consists of all numbers of the form $r \odot 2 = 2^r$, $r \in \mathbb{R}$. It is the entire space V. Hence $\{2\}$ is a basis for V so that $\dim V = 1$.

The coordinate mapping $f : V \to \mathbb{R}$ associated to this basis is given by $f(2^r) = r$ for all $r \in \mathbb{R}$. Equivalently, $f(x) = \log_2 x$, $x \in V$. Notice that $\log_2(x \oplus y) = \log_2 x + \log_2 y$ and $\log_2(r \odot x) = r \log_2 x$.

Vectors $\mathbf{u}_1 = (3, 1)$ and $\mathbf{u}_2 = (2, 1)$ form a basis for $\mathbb{R}^2$.

Problem 1. Find coordinates of the vector $\mathbf{v} = (7, 4)$ with respect to the basis $\mathbf{u}_1, \mathbf{u}_2$.

The desired coordinates x, y satisfy

$$\mathbf{v} = x\mathbf{u}_1 + y\mathbf{u}_2 \iff \begin{cases} 3x + 2y = 7 \\ x + y = 4 \end{cases} \iff \begin{cases} x = -1 \\ y = 5 \end{cases}$$

Problem 2. Find the vector $\mathbf{w}$ whose coordinates with respect to the basis $\mathbf{u}_1, \mathbf{u}_2$ are $(7, 4)$.

$$\mathbf{w} = 7\mathbf{u}_1 + 4\mathbf{u}_2 = 7(3, 1) + 4(2, 1) = (29, 11)$$

Change of coordinates

Given a vector $\mathbf{v} \in \mathbb{R}^2$, let (x, y) be its standard coordinates, i.e., coordinates with respect to the standard basis $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1)$, and let (x', y') be its coordinates with respect to the basis $\mathbf{u}_1 = (3, 1)$, $\mathbf{u}_2 = (2, 1)$.

Problem. Find a relation between (x, y) and (x', y').

By definition, $\mathbf{v} = x\mathbf{e}_1 + y\mathbf{e}_2 = x'\mathbf{u}_1 + y'\mathbf{u}_2$.

In standard coordinates,

$$\begin{pmatrix} x \\ y \end{pmatrix} = x' \begin{pmatrix} 3 \\ 1 \end{pmatrix} + y' \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\implies \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Change of coordinates in $\mathbb{R}^n$

The usual (standard) coordinates of a vector
$\mathbf{v} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ are coordinates relative to the
standard basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$. Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be another
basis for $\mathbb{R}^n$ and $(x_1', x_2', \ldots, x_n')$ be the coordinates of the
same vector $\mathbf{v}$ with respect to this basis. Then

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & \ldots & u_{1n} \\ u_{21} & u_{22} & \ldots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & \ldots & u_{nn} \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix},$$

where the matrix $U = (u_{ij})$ does not depend on the vector $\mathbf{v}$.
Namely, columns of U are coordinates of vectors
$\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ with respect to the standard basis. U is called
the **transition matrix** from the basis $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ to the
standard basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$. The inverse matrix U^{-1} is
called the **transition matrix** from $\mathbf{e}_1, \ldots, \mathbf{e}_n$ to $\mathbf{u}_1, \ldots, \mathbf{u}_n$.

Problem. Find coordinates of the vector $\mathbf{x} = (1, 2, 3)$ with respect to the basis $\mathbf{u}_1 = (1, 1, 0)$, $\mathbf{u}_2 = (0, 1, 1)$, $\mathbf{u}_3 = (1, 1, 1)$.

The nonstandard coordinates (x', y', z') of $\mathbf{x}$ satisfy

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = U \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},$$

where U is the transition matrix from the standard basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to the basis $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

The transition matrix from $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is

$$U_0 = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

The transition matrix from $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ is the inverse matrix: $U = U_0^{-1}$.

The inverse matrix can be computed using row reduction.

$$(U_0 \mid I) = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{pmatrix} = (I \mid U_0^{-1})$$

Thus

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$$

Change of coordinates: general case

Let V be a vector space of dimension n.

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be a basis for V and $g_1 : V \to \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.

Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be another basis for V and $g_2 : V \to \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.

$$
\begin{array}{ccc}
 & V & \\
{\scriptstyle g_1}\swarrow & & \searrow{\scriptstyle g_2} \\
\mathbb{R}^n & \longrightarrow & \mathbb{R}^n
\end{array}
$$
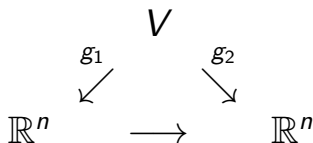
The composition $g_2 \circ g_1^{-1}$ is a transformation of $\mathbb{R}^n$.

It has the form $\mathbf{x} \mapsto U\mathbf{x}$, where U is an $n \times n$ matrix.

U is called the **transition matrix** from $\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_n$ to $\mathbf{u}_1, \mathbf{u}_2 \ldots, \mathbf{u}_n$. Columns of U are coordinates of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ with respect to the basis $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$.