MATH 311

Topics in Applied Mathematics I

# Lecture 32:
# Gradient, divergence, and curl.
# Review of integral calculus.
# Area.

# Gradient, divergence, and curl

**Gradient** of a scalar field $f = f(x_1, x_2 \dots, x_n)$ is

$$\operatorname{grad} f = \Big(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}\Big).$$

**Divergence** of a vector field $\mathbf{F} = (F_1, F_2, \dots, F_n)$ is

$$\operatorname{div} \mathbf{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \cdots + \frac{\partial F_n}{\partial x_n}.$$

**Curl** of a vector field $\mathbf{F} = (F_1, F_2, F_3)$ is

$$\operatorname{curl} \mathbf{F} = \Big(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3},\ \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1},\ \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\Big).$$

Informally, $\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ F_1 & F_2 & F_3 \end{vmatrix}.$

# Del notation

Gradient, divergence, and curl can be denoted in a compact way using the del (a.k.a. nabla a.k.a. atled) "operator"

$$\nabla = \Big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\Big).$$

Namely, $\operatorname{grad} f = \nabla f$, $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F}$, $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F}$.

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**Theorem 1** $\operatorname{div}(\operatorname{curl} \mathbf{F}) = 0$ wherever the vector field $\mathbf{F}$ is twice continuously differentiable.

**Theorem 2** $\operatorname{curl}(\operatorname{grad} f) = \mathbf{0}$ wherever the scalar field $f$ is twice continuously differentiable.

In the del notation, $\nabla \cdot (\nabla \times \mathbf{F}) = 0$ and $\nabla \times (\nabla f) = \mathbf{0}$.

# Riemann sums and Riemann integral

*Definition.* A **Riemann sum** of a function $f : [a, b] \to \mathbb{R}$ with respect to a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ generated by samples $t_j \in [x_{j-1}, x_j]$ is a sum

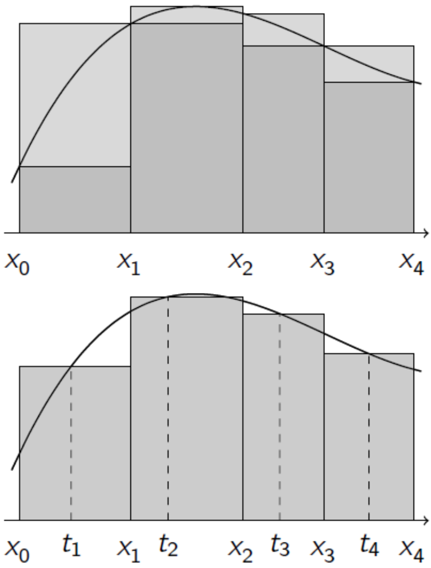
$$\mathcal{S}(f, P, t_j) = \sum_{j=1}^{n} f(t_j)\,(x_j - x_{j-1}).$$

*Remark.* $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a, b]$ if $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. The norm of the partition $P$ is $\|P\| = \max_{1 \le j \le n} |x_j - x_{j-1}|$.

*Definition.* The Riemann sums $\mathcal{S}(f, P, t_j)$ **converge** to a limit $I(f)$ as the norm $\|P\| \to 0$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|P\| < \delta$ implies $|\mathcal{S}(f, P, t_j) - I(f)| < \varepsilon$ for any partition $P$ and choice of samples $t_j$.

If this is the case, then the function $f$ is called **integrable** on $[a, b]$ and the limit $I(f)$ is called the **integral** of $f$ over $[a, b]$, denoted $\int_a^b f(x)\,dx$.

# Riemann sums and Darboux sums



## Integration as a linear operation

**Theorem 1** If functions $f, g$ are integrable on an interval $[a, b]$, then the sum $f + g$ is also integrable on $[a, b]$ and

$$\int_a^b \big(f(x) + g(x)\big)\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx.$$

**Theorem 2** If a function $f$ is integrable on $[a, b]$, then for each $\alpha \in \mathbb{R}$ the scalar multiple $\alpha f$ is also integrable on $[a, b]$ and

$$\int_a^b \alpha f(x)\, dx = \alpha \int_a^b f(x)\, dx.$$

# More properties of integrals

**Theorem** If a function $f$ is integrable on $[a, b]$ and $f([a, b]) \subset [A, B]$, then for each continuous function $g : [A, B] \to \mathbb{R}$ the composition $g \circ f$ is also integrable on $[a, b]$.

**Theorem** If functions $f$ and $g$ are integrable on $[a, b]$, then so is $fg$.

**Theorem** If a function $f$ is integrable on $[a, b]$, then it is integrable on each subinterval $[c, d] \subset [a, b]$. Moreover, for any $c \in (a, b)$ we have

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

# Comparison theorems for integrals

**Theorem 1** If functions $f, g$ are integrable on $[a, b]$ and $f(x) \leq g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

**Theorem 2** If $f$ is integrable on $[a, b]$ and $f(x) \geq 0$ for $x \in [a, b]$, then $\int_a^b f(x)\,dx \geq 0.$

**Theorem 3** If $f$ is integrable on $[a, b]$, then the function $|f|$ is also integrable on $[a, b]$ and

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$$

## Fundamental theorem of calculus

**Theorem** If a function $f$ is continuous on an interval $[a, b]$, then the function

$$F(x) = \int_a^x f(t)\,dt, \quad x \in [a, b],$$

is continuously differentiable on $[a, b]$. Moreover, $F'(x) = f(x)$ for all $x \in [a, b]$.

**Theorem** If a function $F$ is differentiable on $[a, b]$ and the derivative $F'$ is integrable on $[a, b]$, then

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

# Change of the variable in an integral

**Theorem** If $\phi$ is continuously differentiable on a closed, nondegenerate interval $[a, b]$ and $f$ is continuous on $\phi([a, b])$, then

$$\int_{\phi(a)}^{\phi(b)} f(t)\,dt = \int_a^b f(\phi(x))\,\phi'(x)\,dx = \int_a^b f(\phi(x))\,d\phi(x).$$

*Remarks.* • It is possible that $\phi(a) \geq \phi(b)$. To make sense of the integral in this case, we set

$$\int_c^d f(t)\,dt = -\int_d^c f(t)\,dt$$

if $c > d$. Also, we set the integral to be 0 if $c = d$.

• $t = \phi(x)$ is a proper change of the variable only if the function $\phi$ is strictly monotone. However the theorem holds even without this assumption.

# Sets of measure zero

*Definition.* A subset $E$ of the real line $\mathbb{R}$ is said to have **measure zero** if for any $\varepsilon > 0$ the set $E$ can be covered by a sequence of open intervals $J_1, J_2, \dots$ such that $\sum_{n=1}^{\infty} |J_n| < \varepsilon$.

*Examples.* • Any set $E$ that can be represented as a sequence $x_1, x_2, \dots$ (such sets are called **countable**) has measure zero.

Indeed, for any $\varepsilon > 0$, let

$$J_n = \left( x_n - \frac{\varepsilon}{2^{n+1}},\, x_n + \frac{\varepsilon}{2^{n+1}} \right), \quad n = 1, 2, \dots$$

Then $E \subset J_1 \cup J_2 \cup \dots$ and $|J_n| = \varepsilon/2^n$ for all $n \in \mathbb{N}$ so that $\sum_{n=1}^{\infty} |J_n| = \varepsilon$.

• The set $\mathbb{Q}$ of rational numbers has measure zero (since it is countable).

• Nondegenerate interval $[a, b]$ is not a set of measure zero.

# Lebesgue's criterion for Riemann integrability

*Definition.* Suppose $P(x)$ is a property depending on $x \in S$, where $S \subset \mathbb{R}$. We say that $P(x)$ holds for **almost all** $x \in S$ (or **almost everywhere** on $S$) if the set $\{x \in S \mid P(x) \text{ does not hold}\}$ has measure zero.

**Theorem** A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on the interval $[a, b]$ if and only if $f$ is bounded on $[a, b]$ and continuous almost everywhere on $[a, b]$.

# Area

Suppose $\mathcal{P}$ is a nonempty collection of subsets of $\mathbb{R}^2$ such that
**(i)** if $X, Y \in \mathcal{P}$, then $X \cup Y$, $X \cap Y$, $X \setminus Y \in \mathcal{P}$;
**(ii)** if $X \in \mathcal{P}$, then $X + \mathbf{v} \in \mathcal{P}$ for all $\mathbf{v} \in \mathbb{R}^2$.

*Definition.* A function $\mu : \mathcal{P} \to \mathbb{R}$ is called an **area** function if it satisfies the following conditions:

- **(positivity)** $\mu(X) \geq 0$ for all $X \in \mathcal{P}$;
- **(additivity)** $\mu(X \cup Y) = \mu(X) + \mu(Y)$ if $X \cap Y = \emptyset$;
- **(translation invariance)** $\mu(X + \mathbf{v}) = \mu(X)$ for all $X \in \mathcal{P}$ and $\mathbf{v} \in \mathbb{R}^2$;
- $\mu(Q) = 1$, where $Q = [0, 1] \times [0, 1]$ is the unit square.

Any area function satisfies an extra condition:

- **(monotonicity)** $\mu(X) \leq \mu(Y)$ whenever $X \subset Y$.

**Theorem** Let $\mathcal{P}_0$ be the smallest collection of subsets of $\mathbb{R}^2$ that satisfies **(i)** and contains all polygons. Then there exists a unique area function $\mu : \mathcal{P}_0 \to \mathbb{R}$.