Sample problems for the final exam: Solutions

Any problem may be altered or replaced by a different one!

Problem 1 Find the point of intersection of the planes $x+2y-z=1$, $x-3y=-5$, and $2x+y+z=0$ in $\mathbb{R}^3$.

The intersection point (x,y,z) is a solution of the system

$$\begin{cases} x+2y-z=1, \\ x-3y=-5, \\ 2x+y+z=0. \end{cases}$$

To solve the system, we convert its augmented matrix into reduced row echelon form using elementary row operations:

$$\left(\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 1 & -3 & 0 & -5 \\ 2 & 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & -5 & 1 & -6 \\ 2 & 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & -5 & 1 & -6 \\ 0 & -3 & 3 & -2 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & -3 & 3 & -2 \\ 0 & -5 & 1 & -6 \end{array}\right)$$

$$\to \left(\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & \frac{2}{3} \\ 0 & -5 & 1 & -6 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & \frac{2}{3} \\ 0 & 0 & -4 & -\frac{8}{3} \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & \frac{2}{3} \\ 0 & 0 & 1 & \frac{2}{3} \end{array}\right)$$

$$\to \left(\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & 1 & 0 & \frac{4}{3} \\ 0 & 0 & 1 & \frac{2}{3} \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 2 & 0 & \frac{5}{3} \\ 0 & 1 & 0 & \frac{4}{3} \\ 0 & 0 & 1 & \frac{2}{3} \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & \frac{4}{3} \\ 0 & 0 & 1 & \frac{2}{3} \end{array}\right).$$

Thus the three planes intersect at the point $(-1, \frac{4}{3}, \frac{2}{3})$.

Alternative solution: The intersection point (x,y,z) is a solution of the system

$$\begin{cases} x+2y-z=1, \\ x-3y=-5, \\ 2x+y+z=0. \end{cases}$$

Adding all three equations, we obtain $4x=-4$. Hence $x=-1$. Substituting $x=-1$ into the second equation, we obtain $y=\frac{4}{3}$. Substituting $x=-1$ and $y=\frac{4}{3}$ into the third equation, we obtain $z=\frac{2}{3}$. It is easy to check that $x=-1$, $y=\frac{4}{3}$, $z=\frac{2}{3}$ is indeed a solution of the system. Thus $(-1, \frac{4}{3}, \frac{2}{3})$ is the unique intersection point.

Problem 2 Consider a linear operator $L:\mathbb{R}^3\to\mathbb{R}^3$ given by

$$L(\mathbf{v}) = (\mathbf{v}\cdot\mathbf{v}_1)\mathbf{v}_2, \quad \text{where } \mathbf{v}_1=(1,1,1),\ \mathbf{v}_2=(1,2,2).$$

(i) Find the matrix of the operator L.

Given $\mathbf{v} = (x, y, z) \in \mathbb{R}^3$, we have that $\mathbf{v}\cdot\mathbf{v}_1 = x+y+z$ and $L(\mathbf{v}) = (x+y+z, 2(x+y+z), 2(x+y+z))$. Let A denote the matrix of the linear operator L. The columns of A are vectors $L(\mathbf{e}_1), L(\mathbf{e}_2), L(\mathbf{e}_3)$, where $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$ is the standard basis for $\mathbb{R}^3$. Therefore

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}.$$

(ii) Find the dimensions of the range and the kernel of L.

The range $\operatorname{Range}(L)$ of the linear operator L is the subspace of all vectors of the form $L(\mathbf{v})$, where $\mathbf{v} \in \mathbb{R}^3$. It is easy to see that $\operatorname{Range}(L)$ is the line spanned by the vector $\mathbf{v}_2 = (1, 2, 2)$. Hence $\dim \operatorname{Range}(L) = 1$.

The kernel $\ker(L)$ of the operator L is the subspace of all vectors $\mathbf{x} \in \mathbb{R}^3$ such that $L(\mathbf{x}) = \mathbf{0}$. Clearly, $L(\mathbf{x}) = \mathbf{0}$ if and only if $\mathbf{x} \cdot \mathbf{v}_1 = 0$. Therefore $\ker(L)$ is the plane $x + y + z = 0$ orthogonal to $\mathbf{v}_1$ and passing through the origin. Its dimension is 2.

(iii) Find bases for the range and the kernel of L.

Since the range of L is the line spanned by the vector $\mathbf{v}_2 = (1, 2, 2)$, this vector is a basis for the range. The kernel of L is the plane given by the equation $x + y + z = 0$. The general solution of the equation is $x = -t - s$, $y = t$, $z = s$, where $t, s \in \mathbb{R}$. It gives rise to a parametric representation $t(-1, 1, 0) + s(-1, 0, 1)$ of the plane. Thus the kernel of L is spanned by the vectors $(-1, 1, 0)$ and $(-1, 0, 1)$. Since the two vectors are linearly independent, they form a basis for $\ker(L)$.

Problem 3 Let $\mathbf{v}_1 = (1, 1, 1)$, $\mathbf{v}_2 = (1, 1, 0)$, and $\mathbf{v}_3 = (1, 0, 1)$. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator on $\mathbb{R}^3$ such that $L(\mathbf{v}_1) = \mathbf{v}_2$, $L(\mathbf{v}_2) = \mathbf{v}_3$, $L(\mathbf{v}_3) = \mathbf{v}_1$.

(i) Show that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form a basis for $\mathbb{R}^3$.

Let U be a 3×3 matrix such that its columns are vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$:

$$U = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

To find the determinant of U, we subtract the second row from the first one and then expand by the first row:

$$\det U = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1.$$

Since $\det U \neq 0$, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent. It follows that they form a basis for $\mathbb{R}^3$.

(ii) Find the matrix of the operator L relative to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Let A denote the matrix of L relative to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. By definition, the columns of A are coordinates of vectors $L(\mathbf{v}_1), L(\mathbf{v}_2), L(\mathbf{v}_3)$ with respect to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Since $L(\mathbf{v}_1) = \mathbf{v}_2 = 0\mathbf{v}_1 + 1\mathbf{v}_2 + 0\mathbf{v}_3$, $L(\mathbf{v}_2) = \mathbf{v}_3 = 0\mathbf{v}_1 + 0\mathbf{v}_2 + 1\mathbf{v}_3$, $L(\mathbf{v}_3) = \mathbf{v}_1 = 1\mathbf{v}_1 + 0\mathbf{v}_2 + 0\mathbf{v}_3$, we obtain

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

(iii) Find the matrix of the operator L relative to the standard basis.

Let S denote the matrix of L relative to the standard basis for $\mathbb{R}^3$. We have $S = UAU^{-1}$, where A is the matrix of L relative to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ (already found) and U is the transition matrix from $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to the standard basis (the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are consecutive columns of U):

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

To find the inverse U^{-1}, we merge the matrix U with the identity matrix I into one 3×6 matrix and apply row reduction to convert the left half U of this matrix into I. Simultaneously, the right half I will be converted into U^{-1}:

$$(U|I) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right) = (I|U^{-1}).$$

Thus

$$S = UAU^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & -1 & -1 \end{pmatrix}.$$

Alternative solution: Let S denote the matrix of L relative to the standard basis $\mathbf{e}_1 = (1,0,0), \mathbf{e}_2 = (0,1,0), \mathbf{e}_3 = (0,0,1)$. By definition, the columns of S are vectors $L(\mathbf{e}_1), L(\mathbf{e}_2), L(\mathbf{e}_3)$. It is easy to observe that $\mathbf{e}_2 = \mathbf{v}_1 - \mathbf{v}_3$, $\mathbf{e}_3 = \mathbf{v}_1 - \mathbf{v}_2$, and $\mathbf{e}_1 = \mathbf{v}_2 - \mathbf{e}_2 = -\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$. Therefore

$$\begin{aligned} L(\mathbf{e}_1) &= L(-\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) = -L(\mathbf{v}_1) + L(\mathbf{v}_2) + L(\mathbf{v}_3) = -\mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_1 = (1,0,2), \\ L(\mathbf{e}_2) &= L(\mathbf{v}_1 - \mathbf{v}_3) = L(\mathbf{v}_1) - L(\mathbf{v}_3) = \mathbf{v}_2 - \mathbf{v}_1 = (0,0,-1), \\ L(\mathbf{e}_3) &= L(\mathbf{v}_1 - \mathbf{v}_2) = L(\mathbf{v}_1) - L(\mathbf{v}_2) = \mathbf{v}_2 - \mathbf{v}_3 = (0,1,-1). \end{aligned}$$

Thus

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & -1 & -1 \end{pmatrix}.$$

Problem 4 Let $B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

(i) Find all eigenvalues of the matrix B.

The eigenvalues of B are roots of the characteristic equation $\det(B - \lambda I) = 0$. We obtain that

$$\det(B-\lambda I) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^3 - 3(1-\lambda) + 2$$

$$= (1 - 3\lambda + 3\lambda^2 - \lambda^3) - 3(1-\lambda) + 2 = 3\lambda^2 - \lambda^3 = \lambda^2(3-\lambda).$$

Hence the matrix B has two eigenvalues: 0 and 3.

(ii) Find a basis for $\mathbb{R}^3$ consisting of eigenvectors of B.

An eigenvector $\mathbf{x} = (x, y, z)$ of B associated with an eigenvalue λ is a nonzero solution of the vector equation $(B - \lambda I)\mathbf{x} = \mathbf{0}$. First consider the case $\lambda = 0$. We obtain that

$$B\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff x + y + z = 0.$$

The general solution is $x = -t - s$, $y = t$, $z = s$, where $t, s \in \mathbb{R}$. Equivalently, $\mathbf{x} = t(-1, 1, 0) + s(-1, 0, 1)$. Hence the eigenspace of B associated with the eigenvalue 0 is two-dimensional. It is spanned by eigenvectors $\mathbf{v}_1 = (-1, 1, 0)$ and $\mathbf{v}_2 = (-1, 0, 1)$.

Now consider the case $\lambda = 3$. We obtain that

$$(B - 3I)\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} x - z = 0, \\ y - z = 0. \end{cases}$$

The general solution is $x = y = z = t$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_3 = (1, 1, 1)$ is an eigenvector of B associated with the eigenvalue 3.

The vectors $\mathbf{v}_1 = (-1, 1, 0)$, $\mathbf{v}_2 = (-1, 0, 1)$, and $\mathbf{v}_3 = (1, 1, 1)$ are eigenvectors of the matrix B. They are linearly independent since the matrix whose rows are these vectors is nonsingular:

$$\begin{vmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 3 \neq 0.$$

It follows that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is a basis for $\mathbb{R}^3$.

(iii) Find an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of B.

It is easy to check that the vector $\mathbf{v}_3$ is orthogonal to $\mathbf{v}_1$ and $\mathbf{v}_2$. To transform the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ into an orthogonal one, we only need to orthogonalize the pair $\mathbf{v}_1, \mathbf{v}_2$. Using the Gram-Schmidt process, we replace the vector $\mathbf{v}_2$ by

$$\mathbf{u} = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = (-1, 0, 1) - \frac{1}{2}(-1, 1, 0) = (-1/2, -1/2, 1).$$

Now $\mathbf{v}_1, \mathbf{u}, \mathbf{v}_3$ is an orthogonal basis for $\mathbb{R}^3$. Since $\mathbf{u}$ is a linear combination of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, it is also an eigenvector of B associated with the eigenvalue 0.

Finally, vectors $\mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}$, $\mathbf{w}_2 = \frac{\mathbf{u}}{\|\mathbf{u}\|}$, and $\mathbf{w}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|}$ form an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of B. We get that $\|\mathbf{v}_1\| = \sqrt{2}$, $\|\mathbf{u}\| = \sqrt{3/2}$, and $\|\mathbf{v}_3\| = \sqrt{3}$. Thus

$$\mathbf{w}_1 = \frac{1}{\sqrt{2}}(-1,1,0), \quad \mathbf{w}_2 = \frac{1}{\sqrt{6}}(-1,-1,2), \quad \mathbf{w}_3 = \frac{1}{\sqrt{3}}(1,1,1).$$

(iv) Find a diagonal matrix D and an invertible matrix U such that $B = UDU^{-1}$.

The vectors $\mathbf{v}_1 = (-1,1,0)$, $\mathbf{v}_2 = (-1,0,1)$, and $\mathbf{v}_3 = (1,1,1)$ are eigenvectors of the matrix B associated with eigenvalues 0, 0, and 3, respectively. Since these vectors form a basis for $\mathbb{R}^3$, it follows that $B = UDU^{-1}$, where

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \qquad U = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Here U is the transition matrix from the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to the standard basis (its columns are vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$) while D is the matrix of the linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$, $L(\mathbf{x}) = B\mathbf{x}$ with respect to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Problem 5 Let V be a subspace of $\mathbb{R}^4$ spanned by vectors $\mathbf{x}_1 = (1,1,0,0)$, $\mathbf{x}_2 = (2,0,-1,1)$, and $\mathbf{x}_3 = (0,1,1,0)$.

(i) Find the distance from the point $\mathbf{y} = (0,0,0,4)$ to the subspace V.

(ii) Find the distance from the point $\mathbf{y}$ to the orthogonal complement $V^\perp$.

The vector $\mathbf{y}$ is uniquely represented as $\mathbf{y} = \mathbf{p} + \mathbf{o}$, where $\mathbf{p} \in V$ and $\mathbf{o}$ is orthogonal to V, that is, $\mathbf{o} \in V^\perp$. The vector $\mathbf{p}$ is the orthogonal projection of $\mathbf{y}$ onto the subspace V. Since $(V^\perp)^\perp = V$, the vector $\mathbf{o}$ is the orthogonal projection of $\mathbf{y}$ onto the subspace $V^\perp$. It follows that the distance from the point $\mathbf{y}$ to V equals $\|\mathbf{o}\|$ while the distance from $\mathbf{y}$ to $V^\perp$ equals $\|\mathbf{p}\|$.

The orthogonal projection $\mathbf{p}$ of the vector $\mathbf{y}$ onto the subspace V is easily computed when we have an orthogonal basis for V. To get such a basis, we apply the Gram-Schmidt orthogonalization process to the basis $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$:

$$\mathbf{v}_1 = \mathbf{x}_1 = (1,1,0,0), \qquad \mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = (2,0,-1,1) - \frac{2}{2}(1,1,0,0) = (1,-1,-1,1),$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = (0,1,1,0) - \frac{1}{2}(1,1,0,0) - \frac{-2}{4}(1,-1,-1,1) = (0,0,1/2,1/2).$$

Now that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is an orthogonal basis for V we obtain

$$\mathbf{p} = \frac{\mathbf{y} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{y} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 + \frac{\mathbf{y} \cdot \mathbf{v}_3}{\mathbf{v}_3 \cdot \mathbf{v}_3}\mathbf{v}_3 =$$

$$= \frac{0}{2}(1,1,0,0) + \frac{4}{4}(1,-1,-1,1) + \frac{2}{1/2}(0,0,1/2,1/2) = (1,-1,1,3).$$

Consequently, $\mathbf{o} = \mathbf{y} - \mathbf{p} = (0,0,0,4) - (1,-1,1,3) = (-1,1,-1,1)$. Thus the distance from $\mathbf{y}$ to the subspace V equals $\|\mathbf{o}\| = 2$ and the distance from $\mathbf{y}$ to $V^\perp$ equals $\|\mathbf{p}\| = \sqrt{12} = 2\sqrt{3}$.

Problem 6 Consider a vector field $\mathbf{F}(x,y,z) = xyz\mathbf{e}_1 + xy\mathbf{e}_2 + x^2\mathbf{e}_3$.
(i) Find curl($\mathbf{F}$).

$$\text{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & xy & x^2 \end{vmatrix} = \left(\frac{\partial(x^2)}{\partial y} - \frac{\partial(xy)}{\partial z}\right)\mathbf{e}_1 + \left(\frac{\partial(xyz)}{\partial z} - \frac{\partial(x^2)}{\partial x}\right)\mathbf{e}_2$$
$$+ \left(\frac{\partial(xy)}{\partial x} - \frac{\partial(xyz)}{\partial y}\right)\mathbf{e}_3 = (xy - 2x)\mathbf{e}_2 + (y - xz)\mathbf{e}_3.$$

(ii) Find the integral of the vector field curl($\mathbf{F}$) along a hemisphere $H = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1,\ z \geq 0\}$. Orient the hemisphere by the normal vector $\mathbf{n} = (0,0,1)$ at the point $(0,0,1)$.

According to Stokes' Theorem,

$$\iint_H \text{curl}(\mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial H} \mathbf{F} \cdot d\mathbf{s},$$

where the boundary ∂H is oriented consistently with H. The boundary is a circle, $\partial H = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1,\ z = 0\}$. It is parametrized (with the right orientation) by a path $\mathbf{x} : [0, 2\pi] \to \mathbb{R}^3$, $\mathbf{x}(t) = (\cos t, \sin t, 0)$. We have $\mathbf{F}(\mathbf{x}(t)) = (0, \cos t\ \sin t, \cos^2 t)$ and $\mathbf{x}'(t) = (-\sin t, \cos t, 0)$. Therefore

$$\oint_{\partial H} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt = \int_0^{2\pi} \cos^2 t\ \sin t\, dt = -\frac{1}{3}\cos^3 t\ \Big|_{t=0}^{2\pi} = 0.$$

Problem 7 Find the volume of a parallelepiped bounded by planes $x + 2y - z = -1$, $x + 2y - z = 1$, $x - 3y = -5$, $x - 3y = 0$, $2x + y + z = 0$, and $2x + y + z = 2$.

Let P denote the parallelepiped. The volume of P can be found as a triple integral:

$$\text{Volume}(P) = \iiint_P 1\, dx\, dy\, dz.$$

To evaluate the integral, we are going to change variables. New variables are $u = x+2y-z$, $v = x-3y$, and $w = 2x + y + z$. In these variables the parallelepiped P is given by $-1 \leq u \leq 1$, $-5 \leq v \leq 0$, $0 \leq w \leq 2$. It follows that

$$\text{Volume}(P) = \int_0^2 \int_{-5}^0 \int_{-1}^1 \left|\det \frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\, dv\, dw.$$

Our change of coordinates is linear,

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 1 & -3 & 0 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Let U denote the above matrix. The Jacobian matrix $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$ equals U at every point of $\mathbb{R}^3$. Consequently, the Jacobian matrix $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$ equals U^{-1} everywhere on $\mathbb{R}^3$. We obtain

$$\det U = \begin{vmatrix} 1 & 2 & -1 \\ 1 & -3 & 0 \\ 2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 3 & 0 \\ 1 & -3 & 0 \\ 2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 3 \\ 1 & -3 \end{vmatrix} = -12.$$

Hence $\det(U^{-1}) = (\det U)^{-1} = -1/12$. Then

$$\text{Volume}(P) = \int_0^2 \int_{-5}^0 \int_{-1}^1 \left|\det(U^{-1})\right| \, du\, dv\, dw = \frac{1}{12} \cdot 2 \cdot 5 \cdot 2 = \frac{5}{3}.$$