

MATH 311

Topics in Applied Mathematics I

**Lecture 4:**

**Applications of systems of linear equations.  
Matrix algebra.**

## Applications of systems of linear equations

**Problem 1.** Find the point of intersection of the lines  $x - y = -2$  and  $2x + 3y = 6$  in  $\mathbb{R}^2$ .

$$\begin{cases} x - y = -2 \\ 2x + 3y = 6 \end{cases}$$

**Problem 2.** Find the point of intersection of the planes  $x - y = 2$ ,  $2x - y - z = 3$ , and  $x + y + z = 6$  in  $\mathbb{R}^3$ .

$$\begin{cases} x - y = 2 \\ 2x - y - z = 3 \\ x + y + z = 6 \end{cases}$$

*Method of undetermined coefficients* often involves solving systems of linear equations.

**Problem 3.** Find a quadratic polynomial  $p(x)$  such that  $p(1) = 4$ ,  $p(2) = 3$ , and  $p(3) = 4$ .

Suppose that  $p(x) = ax^2 + bx + c$ . Then  
 $p(1) = a + b + c$ ,  $p(2) = 4a + 2b + c$ ,  
 $p(3) = 9a + 3b + c$ .

$$\begin{cases} a + b + c = 4 \\ 4a + 2b + c = 3 \\ 9a + 3b + c = 4 \end{cases}$$

*Method of undetermined coefficients* often involves solving systems of linear equations.

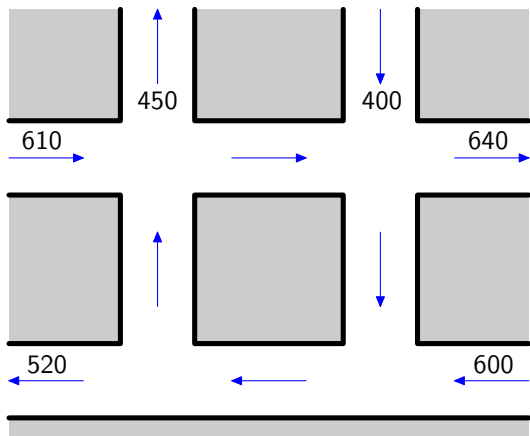
**Problem 3.** Find a quadratic polynomial  $p(x)$  such that  $p(1) = 4$ ,  $p(2) = 3$ , and  $p(3) = 4$ .

*Alternative choice of coefficients:*  $p(x) = \tilde{a} + \tilde{b}x + \tilde{c}x^2$ .

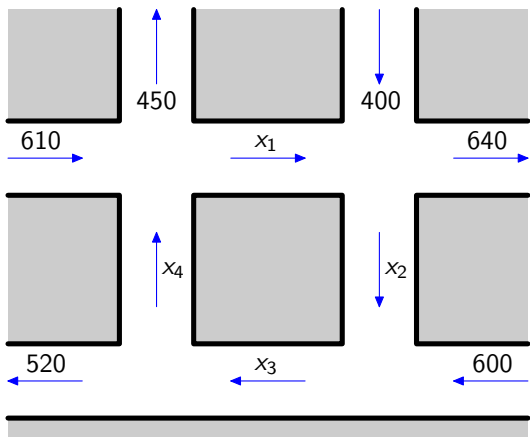
Then  $p(1) = \tilde{a} + \tilde{b} + \tilde{c}$ ,  $p(2) = \tilde{a} + 2\tilde{b} + 4\tilde{c}$ ,  
 $p(3) = \tilde{a} + 3\tilde{b} + 9\tilde{c}$ .

$$\begin{cases} \tilde{a} + \tilde{b} + \tilde{c} = 4 \\ \tilde{a} + 2\tilde{b} + 4\tilde{c} = 3 \\ \tilde{a} + 3\tilde{b} + 9\tilde{c} = 4 \end{cases}$$

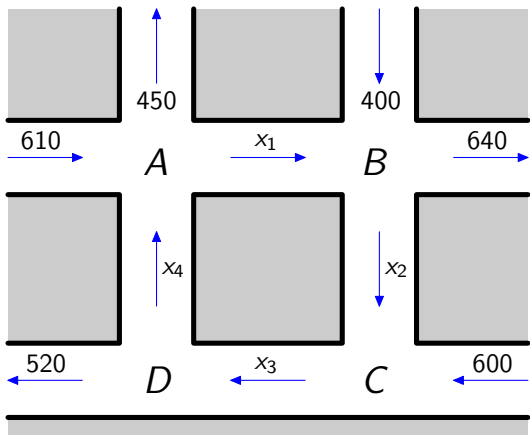
## Traffic flow



**Problem.** Determine the amount of traffic between each of the four intersections.



$$x_1 = ?, \quad x_2 = ?, \quad x_3 = ?, \quad x_4 = ?$$



At each intersection, the incoming traffic has to match the outgoing traffic.

$$\text{Intersection } A: \quad x_4 + 610 = x_1 + 450$$

$$\text{Intersection } B: \quad x_1 + 400 = x_2 + 640$$

$$\text{Intersection } C: \quad x_2 + 600 = x_3$$

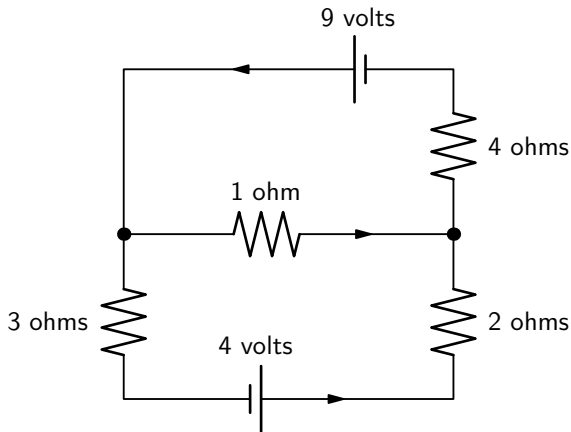
$$\text{Intersection } D: \quad x_3 = x_4 + 520$$

$$\begin{cases} x_4 + 610 = x_1 + 450 \\ x_1 + 400 = x_2 + 640 \\ x_2 + 600 = x_3 \\ x_3 = x_4 + 520 \end{cases}$$

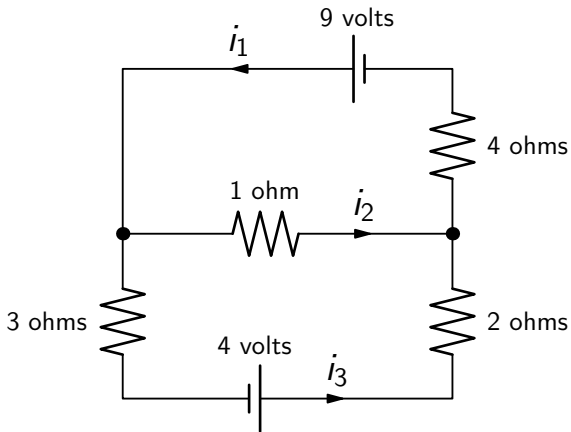
$$\iff \begin{cases} -x_1 + x_4 = -160 \\ x_1 - x_2 = 240 \\ x_2 - x_3 = -600 \\ x_3 - x_4 = 520 \end{cases}$$



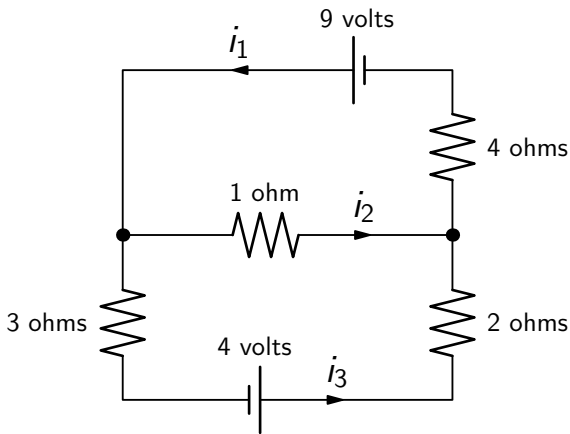
## Electrical network



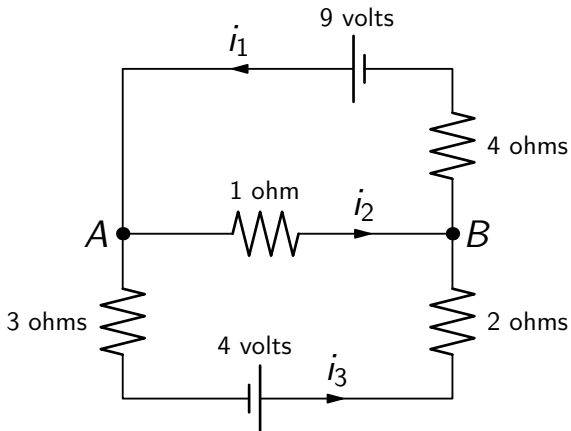
**Problem.** Determine the amount of current in each branch of the network.



$$i_1 = ?, \quad i_2 = ?, \quad i_3 = ?$$



**Kirchhof's law #1 (junction rule):** at every node the sum of the incoming currents equals the sum of the outgoing currents.



Node A:  $i_1 = i_2 + i_3$

Node B:  $i_2 + i_3 = i_1$

## Electrical network

**Kirchhof's law #2 (loop rule):** around every loop the algebraic sum of all voltages is zero.

**Ohm's law:** for every resistor the voltage drop  $E$ , the current  $i$ , and the resistance  $R$  satisfy  $E = iR$ .

$$\text{Top loop:} \quad 9 - i_2 - 4i_1 = 0$$

$$\text{Bottom loop:} \quad 4 - 2i_3 + i_2 - 3i_3 = 0$$

$$\text{Big loop:} \quad 4 - 2i_3 - 4i_1 + 9 - 3i_3 = 0$$

*Remark.* The 3rd equation is the sum of the first two equations.

$$\begin{cases} i_1 = i_2 + i_3 \\ 9 - i_2 - 4i_1 = 0 \\ 4 - 2i_3 + i_2 - 3i_3 = 0 \end{cases}$$

$$\iff \begin{cases} i_1 - i_2 - i_3 = 0 \\ 4i_1 + i_2 = 9 \\ -i_2 + 5i_3 = 4 \end{cases}$$

## Matrices

*Definition.* An **m-by-n matrix** is a rectangular array of numbers that has  $m$  rows and  $n$  columns:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

*Notation:*  $A = (a_{ij})_{1 \leq i \leq n, 1 \leq j \leq m}$  or simply  $A = (a_{ij})$  if the dimensions are known.

## Vector algebra

Let  $\mathbf{a} = (a_1, a_2, \dots, a_n)$  and  $\mathbf{b} = (b_1, b_2, \dots, b_n)$  be  $n$ -dimensional vectors, and  $r \in \mathbb{R}$  be a scalar.

*Vector sum:*  $\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$

*Scalar multiple:*  $r\mathbf{a} = (ra_1, ra_2, \dots, ra_n)$

*Zero vector:*  $\mathbf{0} = (0, 0, \dots, 0)$

*Negative of a vector:*  $-\mathbf{b} = (-b_1, -b_2, \dots, -b_n)$

*Vector difference:*

$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) = (a_1 - b_1, a_2 - b_2, \dots, a_n - b_n)$



Given  $n$ -dimensional vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  and scalars  $r_1, r_2, \dots, r_k$ , the expression

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \dots + r_k\mathbf{v}_k$$

is called a **linear combination** of vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ .

Also, *vector addition* and *scalar multiplication* are called **linear operations**.

## Matrix algebra

*Definition.* Let  $A = (a_{ij})$  and  $B = (b_{ij})$  be  $m \times n$  matrices. The **sum**  $A + B$  is defined to be the  $m \times n$  matrix  $C = (c_{ij})$  such that  $c_{ij} = a_{ij} + b_{ij}$  for all indices  $i, j$ .

That is, two matrices with the same dimensions can be added by adding their corresponding entries.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \\ a_{31} + b_{31} & a_{32} + b_{32} \end{pmatrix}$$

*Definition.* Given an  $m \times n$  matrix  $A = (a_{ij})$  and a number  $r$ , the **scalar multiple**  $rA$  is defined to be the  $m \times n$  matrix  $D = (d_{ij})$  such that  $d_{ij} = ra_{ij}$  for all indices  $i, j$ .

That is, to multiply a matrix by a scalar  $r$ , one multiplies each entry of the matrix by  $r$ .

$$r \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} ra_{11} & ra_{12} & ra_{13} \\ ra_{21} & ra_{22} & ra_{23} \\ ra_{31} & ra_{32} & ra_{33} \end{pmatrix}$$

The  $m \times n$  **zero matrix** (all entries are zeros) is denoted  $O_{mn}$  or simply  $O$ .

**Negative** of a matrix:  $-A$  is defined as  $(-1)A$ .

Matrix **difference**:  $A - B$  is defined as  $A + (-B)$ .

As far as the *linear operations* (addition and scalar multiplication) are concerned, the  $m \times n$  matrices can be regarded as  $mn$ -dimensional vectors.

## Examples

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

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$$A + B = \begin{pmatrix} 5 & 2 & 0 \\ 1 & 2 & 2 \end{pmatrix}, \quad A - B = \begin{pmatrix} 1 & 2 & -2 \\ 1 & 0 & 0 \end{pmatrix},$$

$$2C = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}, \quad 3D = \begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix},$$

$$2C + 3D = \begin{pmatrix} 7 & 3 \\ 0 & 5 \end{pmatrix}, \quad A + D \text{ is not defined.}$$

## Properties of linear operations

$$(A + B) + C = A + (B + C)$$

$$A + B = B + A$$

$$A + O = O + A = A$$

$$A + (-A) = (-A) + A = O$$

$$r(sA) = (rs)A$$

$$r(A + B) = rA + rB$$

$$(r + s)A = rA + sA$$

$$1A = A$$

$$0A = O$$