MATH 311

Topics in Applied Mathematics I

Lecture 13: Review for Test 1.

Topics for Test 1

Part I: Elementary linear algebra (Leon/Colley 1.1–1.5, 2.1–2.2)

- Systems of linear equations: elementary operations, Gaussian elimination, back substitution.
- Matrix of coefficients and augmented matrix. Elementary row operations, row echelon form and reduced row echelon form.
- Matrix algebra. Inverse matrix.
- Determinants: explicit formulas for 2×2 and 3×3 matrices, row and column expansions, elementary row and column operations.

Topics for Test 1

Part II: Abstract linear algebra (Leon/Colley 3.1–3.2)

- Vector spaces (coordinate vectors, matrices, polynomials, functional spaces).
- Basic properties of vector spaces.
- Subspaces of vector spaces.
- Span, spanning set.

Sample problems for Test 1

Problem 1 Find a quadratic polynomial $p(x)$ such that $p(1) = 1$, $p(2) = 3$, and $p(3) = 7$.

Problem 2 Let A be a square matrix such that $A^3 = O$.
(i) Prove that the matrix A is not invertible.
(ii) Prove that the matrix $A + I$ is invertible.

Problem 3 Let $A = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}$.

(i) Evaluate the determinant of the matrix A.
(ii) Find the inverse matrix A^{-1}.

Sample problems for Test 1

Problem 4 Solve an equation $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & x \\ 4 & 9 & x^2 \end{vmatrix} = 0.$

Problem 5 Determine which of the following subsets of $\mathbb{R}^3$ are subspaces. Briefly explain.

(i) The set S_1 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $xyz = 0$.
(ii) The set S_2 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $x + y + z = 0$.
(iii) The set S_3 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $y^2 + z^2 = 0$.
(iv) The set S_4 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $y^2 - z^2 = 0$.

Problem 6 Let V denote the solution set of a system

$$\begin{cases} x_2 + 2x_3 + 3x_4 = 0, \\ x_1 + 2x_2 + 3x_3 + 4x_4 = 0. \end{cases}$$

Find a finite spanning set for this subspace of $\mathbb{R}^4$.

Problem 1. Find a quadratic polynomial $p(x)$ such that $p(1) = 1$, $p(2) = 3$, and $p(3) = 7$.

Let $p(x) = a + bx + cx^2$. Then $p(1) = a + b + c$, $p(2) = a + 2b + 4c$, and $p(3) = a + 3b + 9c$.
The coefficients a, b, and c have to be chosen so that

$$\begin{cases} a + b + c = 1, \\ a + 2b + 4c = 3, \\ a + 3b + 9c = 7. \end{cases}$$

We solve this system of linear equations using elementary operations:

$$\begin{cases} a + b + c = 1 \\ a + 2b + 4c = 3 \\ a + 3b + 9c = 7 \end{cases} \iff \begin{cases} a + b + c = 1 \\ b + 3c = 2 \\ a + 3b + 9c = 7 \end{cases}$$

$$\iff \begin{cases} a+b+c=1 \\ b+3c=2 \\ a+3b+9c=7 \end{cases} \iff \begin{cases} a+b+c=1 \\ b+3c=2 \\ 2b+8c=6 \end{cases}$$

$$\iff \begin{cases} a+b+c=1 \\ b+3c=2 \\ b+4c=3 \end{cases} \iff \begin{cases} a+b+c=1 \\ b+3c=2 \\ c=1 \end{cases}$$

$$\iff \begin{cases} a+b+c=1 \\ b=-1 \\ c=1 \end{cases} \iff \begin{cases} a=1 \\ b=-1 \\ c=1 \end{cases}$$

Thus the desired polynomial is $p(x) = x^2 - x + 1$.

Problem 2 Let A be a square matrix such that $A^3 = O$.

(i) Prove that the matrix A is not invertible.

The proof is by contradiction. Assume that A is invertible. Since any product of invertible matrices is also invertible, the matrix $A^3 = AAA$ should be invertible as well. However $A^3 = O$ is singular.

Problem 2 Let A be a square matrix such that $A^3 = O$.

(ii) Prove that the matrix $A + I$ is invertible.

It is enough to show that the equation $(A + I)\mathbf{x} = \mathbf{0}$ (where $\mathbf{x}$ and $\mathbf{0}$ are column vectors) has a unique solution $\mathbf{x} = \mathbf{0}$.
Indeed, $(A+I)\mathbf{x} = \mathbf{0} \implies A\mathbf{x} + I\mathbf{x} = \mathbf{0} \implies A\mathbf{x} = -\mathbf{x}$.
Then $A^2\mathbf{x} = A(A\mathbf{x}) = A(-\mathbf{x}) = -A\mathbf{x} = -(-\mathbf{x}) = \mathbf{x}$.
Further, $A^3\mathbf{x} = A(A^2\mathbf{x}) = A\mathbf{x} = -\mathbf{x}$. On the other hand, $A^3\mathbf{x} = O\mathbf{x} = \mathbf{0}$. Hence $-\mathbf{x} = \mathbf{0} \implies \mathbf{x} = \mathbf{0}$.

Alternatively, we can use equalities

$$X^3 + Y^3 = (X+Y)(X^2 - XY + Y^2) = (X^2 - XY + Y^2)(X+Y),$$

which hold whenever matrices X and Y commute: $XY = YX$.
In particular, they hold for $X = A$ and $Y = I$. We obtain

$$(A+I)(A^2 - A + I) = (A^2 - A + I)(A + I) = A^3 + I^3 = I$$

so that $(A+I)^{-1} = A^2 - A + I$.

Problem 3. Let $A = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}$.

(i) Evaluate the determinant of the matrix A.

Subtract the 4th row of A from the 3rd row:

$$\begin{vmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 1 \end{vmatrix}.$$

Expand the determinant by the 3rd row:

$$\begin{vmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 1 \end{vmatrix} = (-1)\begin{vmatrix} 1 & -2 & 1 \\ 2 & 3 & 0 \\ 2 & 0 & 1 \end{vmatrix}.$$

Expand the determinant by the 3rd column:

$$(-1)\begin{vmatrix} 1 & -2 & 1 \\ 2 & 3 & 0 \\ 2 & 0 & 1 \end{vmatrix} = (-1)\left(\begin{vmatrix} 2 & 3 \\ 2 & 0 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix}\right) = -1.$$

Problem 3. Let $A = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}$.

(ii) Find the inverse matrix A^{-1}.

First we merge the matrix A with the identity matrix into one 4×8 matrix

$$(A \,|\, I) = \left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 2 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right).$$

Then we apply elementary row operations to this matrix until the left part becomes the identity matrix.

Subtract 2 times the 1st row from the 2nd row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 2 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

Subtract 2 times the 1st row from the 3rd row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 0 & 4 & -9 & -1 & -2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

Subtract 2 times the 1st row from the 4th row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 0 & 4 & -9 & -1 & -2 & 0 & 1 & 0 \\ 0 & 4 & -8 & -1 & -2 & 0 & 0 & 1 \end{array}\right)$$

Subtract 2 times the 4th row from the 2nd row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 4 & -9 & -1 & -2 & 0 & 1 & 0 \\ 0 & 4 & -8 & -1 & -2 & 0 & 0 & 1 \end{array}\right)$$

Subtract the 4th row from the 3rd row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 4 & -8 & -1 & -2 & 0 & 0 & 1 \end{array}\right)$$

Add 4 times the 2nd row to the 4th row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 32 & -1 & 6 & 4 & 0 & -7 \end{array}\right)$$

Add 32 times the 3rd row to the 4th row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right)$$

Add 10 times the 3rd row to the 2nd row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right)$$

Add the 4th row to the 1st row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 0 & 7 & 4 & 32 & -39 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right)$$

Add 4 times the 3rd row to the 1st row:

$$\left(\begin{array}{cccc|cccc} 1 & -2 & 0 & 0 & 7 & 4 & 36 & -43 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right)$$

Subtract 2 times the 2nd row from the 1st row:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3 & 2 & 16 & -19 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right)$$

Multiply the 2nd, the 3rd, and the 4th rows by -1:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3 & 2 & 16 & -19 \\ 0 & 1 & 0 & 0 & -2 & -1 & -10 & 12 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -6 & -4 & -32 & 39 \end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3 & 2 & 16 & -19 \\ 0 & 1 & 0 & 0 & -2 & -1 & -10 & 12 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -6 & -4 & -32 & 39 \end{array}\right) = (I \mid A^{-1})$$

Finally the left part of our 4×8 matrix is transformed into the identity matrix. Therefore the current right part is the inverse matrix of A. Thus

$$A^{-1} = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 2 & 16 & -19 \\ -2 & -1 & -10 & 12 \\ 0 & 0 & -1 & 1 \\ -6 & -4 & -32 & 39 \end{pmatrix}.$$

Problem 3. Let $A = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}$.

(i) Evaluate the determinant of the matrix A.

Alternative solution: We have transformed A into the identity matrix using elementary row operations. These included no row exchanges and three row multiplications, each time by -1.

It follows that $\det I = (-1)^3 \det A$.
$\implies \quad \det A = -\det I = -1$.

Problem 4. Solve an equation $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & x \\ 4 & 9 & x^2 \end{vmatrix} = 0.$

Let us evaluate the determinant using row reduction and column expansion:

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & x \\ 4 & 9 & x^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & x-2 \\ 4 & 9 & x^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & x-2 \\ 0 & 5 & x^2-4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x-2 \\ 5 & x^2-4 \end{vmatrix} = x^2 - 4 - 5(x-2) = x^2 - 5x + 6.$$

Hence our equation is quadratic. The solutions are $x_{1,2} = \dfrac{5 \pm \sqrt{5^2 - 4 \cdot 6}}{2} = \dfrac{5 \pm 1}{2}$. That is, $x_1 = 2$, $x_2 = 3$.

Problem 4. Solve an equation $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & x \\ 4 & 9 & x^2 \end{vmatrix} = 0.$

Alternative solution: It is easy to observe that $x = 2$ and $x = 3$ are solutions (for each of these values, the matrix has two identical columns). To show that there are no more solutions, we expand the determinant by the 3rd column:

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & x \\ 4 & 9 & x^2 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix} - x \begin{vmatrix} 1 & 1 \\ 4 & 9 \end{vmatrix} + x^2 \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}.$$

Since $\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1 \neq 0$, our equation is quadratic. Therefore it has at most two solutions.

The determinant in the last problem is an example of the Vandermonde determinant

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1 \le i < j \le n} (x_j - x_i).$$

Problem 5. Determine which of the following subsets of $\mathbb{R}^3$ are subspaces. Briefly explain.

A subset of $\mathbb{R}^3$ is a subspace if it is closed under addition and scalar multiplication. Besides, the subset must not be empty.

(i) The set S_1 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $xyz = 0$.

$(0, 0, 0) \in S_1 \implies S_1$ is not empty.

$xyz = 0 \implies (rx)(ry)(rz) = r^3 xyz = 0.$

That is, $\mathbf{v} = (x, y, z) \in S_1 \implies r\mathbf{v} = (rx, ry, rz) \in S_1.$

Hence S_1 is closed under scalar multiplication.

However S_1 is not closed under addition.

Counterexample: $(1, 1, 0) + (0, 0, 1) = (1, 1, 1).$

Problem 5. Determine which of the following subsets of $\mathbb{R}^3$ are subspaces. Briefly explain.

A subset of $\mathbb{R}^3$ is a subspace if it is closed under addition and scalar multiplication. Besides, the subset must not be empty.

(ii) The set S_2 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $x + y + z = 0$.

$(0, 0, 0) \in S_2 \implies S_2$ is not empty.

$x + y + z = 0 \implies rx + ry + rz = r(x + y + z) = 0.$
Hence S_2 is closed under scalar multiplication.

$x + y + z = x' + y' + z' = 0 \implies$
$(x + x') + (y + y') + (z + z') = (x + y + z) + (x' + y' + z') = 0.$
That is, $\mathbf{v} = (x, y, z),\ \mathbf{v}' = (x', y', z') \in S_2$
$\implies \mathbf{v} + \mathbf{v}' = (x + x', y + y', z + z') \in S_2.$
Hence S_2 is closed under addition.

(iii) The set S_3 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $y^2 + z^2 = 0$.

$y^2 + z^2 = 0 \iff y = z = 0.$

S_3 is a nonempty set closed under addition and scalar multiplication.

(iv) The set S_4 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $y^2 - z^2 = 0$.

S_4 is a nonempty set closed under scalar multiplication.
However S_4 is not closed under addition.
Counterexample: $(0, 1, 1) + (0, 1, -1) = (0, 2, 0)$.

Problem 6 Let V denote the solution set of a system

$$\begin{cases} x_2 + 2x_3 + 3x_4 = 0, \\ x_1 + 2x_2 + 3x_3 + 4x_4 = 0. \end{cases}$$

Find a finite spanning set for this subspace of $\mathbb{R}^4$.

To find a spanning set, we need to solve the system. To this end, we subtract 2 times the 1st equation from the 2nd one, then switch the equations:

$$\begin{cases} x_1 - x_3 - 2x_4 = 0 \\ x_2 + 2x_3 + 3x_4 = 0 \end{cases} \iff \begin{cases} x_1 = x_3 + 2x_4 \\ x_2 = -2x_3 - 3x_4 \end{cases}$$

x_3 and x_4 are free variables. General solution:

$$\begin{cases} x_1 = t + 2s \\ x_2 = -2t - 3s \\ x_3 = t \\ x_4 = s \end{cases} \qquad (t, s \in \mathbb{R})$$

In vector form, $(x_1, x_2, x_3, x_4) = t(1, -2, 1, 0) + s(2, -3, 0, 1)$. We conclude that the solution set is spanned by vectors $(1, -2, 1, 0)$ and $(2, -3, 0, 1)$.