MATH 311

Topics in Applied Mathematics I

# Lecture 33: Line integrals. Green's theorem.

# Path

*Definition.* A **path** in $\mathbb{R}^n$ is a continuous function $\mathbf{x} : [a, b] \to \mathbb{R}^n$.

Paths provide parametrizations for curves.

Length of the path $\mathbf{x}$ is defined as $L = \sup_P \sum_{j=1}^{k} \|\mathbf{x}(t_j) - \mathbf{x}(t_{j-1})\|$ over all partitions $P = \{t_0, t_1, \dots, t_k\}$ of the interval $[a, b]$.

**Theorem** The length of a smooth path $\mathbf{x} : [a, b] \to \mathbb{R}^n$ is $\displaystyle\int_a^b \|\mathbf{x}'(t)\| \, dt$.

Arclength parameter: $\displaystyle s(t) = \int_a^t \|\mathbf{x}'(\tau)\| \, d\tau$.

# Scalar line integral

Scalar line integral is an integral of a scalar function $f$ over a path $\mathbf{x} : [a, b] \to \mathbb{R}^n$ of finite length relative to the arclength. It is defined as a limit of Riemann sums

$$\mathcal{S}(f, P, \tau_j) = \sum_{j=1}^{k} f(\mathbf{x}(\tau_j)) \big(s(t_j) - s(t_{j-1})\big),$$

where $P = \{t_0, t_1, \dots, t_k\}$ is a partition of $[a, b]$, $\tau_j \in [t_j, t_{j-1}]$ for $1 \le j \le k$, and $s$ is the arclength parameter of the path $\mathbf{x}$.

**Theorem** Let $\mathbf{x} : [a, b] \to \mathbb{R}^n$ be a smooth path and $f$ be a function defined on the image of this path. Then

$$\int_{\mathbf{x}} f\, ds = \int_a^b f(\mathbf{x}(t))\, \|\mathbf{x}'(t)\|\, dt.$$

$ds$ is referred to as the arclength element.

## Vector line integral

Vector line integral is an integral of a vector field over a smooth path. It is a scalar.

*Definition.* Let $\mathbf{x} : [a, b] \to \mathbb{R}^n$ be a smooth path and $\mathbf{F}$ be a vector field defined on the image of this path. Then $\displaystyle\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt.$

Alternatively, the integral of $\mathbf{F}$ over $\mathbf{x}$ can be represented as the integral of a **differential form**

$$\int_{\mathbf{x}} F_1\, dx_1 + F_2\, dx_2 + \cdots + F_n\, dx_n,$$

where $\mathbf{F} = (F_1, F_2, \ldots, F_n)$ and $dx_i = x_i'(t)\, dt$.

# Applications of line integrals

- Mass of a wire

If $f$ is the density on a wire $C$, then $\int_C f\, ds$ is the mass of $C$.

- Work of a force

If $\mathbf{F}$ is a force field, then $\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s}$ is the work done by $\mathbf{F}$ on a particle that moves along the path $\mathbf{x}$.

- Circulation of fluid

If $\mathbf{F}$ is the velocity field of a planar fluid, then the circulation of the fluid across a closed curve $C$ is $\oint_C \mathbf{F} \cdot d\mathbf{s}$.

- Flux of fluid

If $\mathbf{F}$ is the velocity field of a planar fluid, then the flux of the fluid across a closed curve $C$ is $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds$, where $\mathbf{n}$ is the outward unit normal vector to $C$.

# Line integrals and reparametrization

Given a path $\mathbf{x} : [a, b] \to \mathbb{R}^n$, we say that another path $\mathbf{y} : [c, d] \to \mathbb{R}^n$ is a **reparametrization** of $\mathbf{x}$ if there exists a continuous invertible function $u : [c, d] \to [a, b]$ such that $\mathbf{y}(t) = \mathbf{x}(u(t))$ for all $t \in [c, d]$.

The reparametrization may be orientation-preserving (when $u$ is increasing) or orientation-reversing (when $u$ is decreasing).

**Theorem 1** Any scalar line integral is invariant under reparametrizations.

**Theorem 2** Any vector line integral is invariant under orientation-preserving reparametrizations and changes its sign under orientation-reversing reparametrizations.

As a consequence, we can define the integral of a function over a simple curve and the integral of a vector field over a simple oriented curve.

# Green's Theorem

**Theorem** Let $D \subset \mathbb{R}^2$ be a closed, bounded region with piecewise smooth boundary $\partial D$ oriented so that $D$ is on the left as one traverses $\partial D$. Then for any smooth vector field $\mathbf{F} = (M, N)$ on $D$,

$$\oint_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \iint_D \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy$$

or, equivalently,

$$\oint_{\partial D} M\, dx + N\, dy = \iint_D \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy.$$

# Examples

Consider vector fields $\mathbf{F}(x, y) = (-y, 0)$, $\mathbf{G}(x, y) = (0, x)$, and $\mathbf{H}(x, y) = (y, x)$.

According to Green's Theorem,

$$\oint_{\partial D} -y\,dx = \iint_D 1\,dx\,dy = \mathrm{area}(D),$$

$$\oint_{\partial D} x\,dy = \iint_D 1\,dx\,dy = \mathrm{area}(D),$$

$$\oint_{\partial D} y\,dx + x\,dy = \iint_D 0\,dx\,dy = 0.$$

# Green's Theorem

*Proof in the case* $D = [0,1] \times [0,1]$ *and* $\mathbf{F} = (0, N)$*:*

$$\int_0^1 \frac{\partial N}{\partial x}(\xi, y)\, d\xi = N(1,y) - N(0,y)$$

for any $y \in [0,1]$ due to the Fundamental Theorem of Calculus. Integrating this equality by $y$ over $[0,1]$, we obtain

$$\iint_D \frac{\partial N}{\partial x}\, dx\, dy = \int_0^1 N(1,y)\, dy - \int_0^1 N(0,y)\, dy.$$

Let $P_1 = (0,0)$, $P_2 = (1,0)$, $P_3 = (1,1)$, and $P_4 = (0,1)$. The first integral in the right-hand side equals the vector integral of the field $\mathbf{F}$ over the segment $P_2P_3$. The second integral equals the integral of $\mathbf{F}$ over the segment $P_1P_4$. Also, the integral of $\mathbf{F}$ over any horizontal segment is 0. It follows that the entire right-hand side equals the integral of $\mathbf{F}$ over the broken line $P_1P_2P_3P_4P_1$, that is, over $\partial D$.

# Divergence Theorem

**Theorem** Let $D \subset \mathbb{R}^2$ be a closed, bounded region with piecewise smooth boundary $\partial D$ oriented so that $D$ is on the left as one traverses $\partial D$. Then for any smooth vector field $\mathbf{F}$ on $D$,

$$\oint_{\partial D} \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \nabla \cdot \mathbf{F}\, dA.$$

*Proof:* Let $\mathcal{L}$ denote the rotation of the plane $\mathbb{R}^2$ by $90^{\mathrm{o}}$ about the origin (counterclockwise). $\mathcal{L}$ is a linear transformation preserving the dot product. Therefore

$$\oint_{\partial D} \mathbf{F} \cdot \mathbf{n}\, ds = \oint_{\partial D} \mathcal{L}(\mathbf{F}) \cdot \mathcal{L}(\mathbf{n})\, ds.$$

Note that $\mathcal{L}(\mathbf{n})$ is the unit tangent vector to $\partial D$. It follows that the right-hand side is the vector integral of $\mathcal{L}(\mathbf{F})$ over $\partial D$. If $\mathbf{F} = (M, N)$ then $\mathcal{L}(\mathbf{F}) = (-N, M)$. By Green's Theorem,

$$\oint_{\partial D} \mathcal{L}(\mathbf{F}) \cdot d\mathbf{s} = \oint_{\partial D} -N\, dx + M\, dy = \iint_D \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx\, dy.$$