MATH 311

Topics in Applied Mathematics I

# Lecture 35:
# Gauss' theorem.
# Stokes' theorem.

# Surface integrals

Let $\mathbf{X} : D \to \mathbb{R}^3$ be a smooth parametrized surface, where $D \subset \mathbb{R}^2$ is a bounded region. Then for any continuous function $f : \mathbf{X}(D) \to \mathbb{R}$, the **scalar integral** of $f$ over the surface $\mathbf{X}$ is

$$\iint_{\mathbf{X}} f\, dS = \iint_D f\left(\mathbf{X}(s,t)\right) \left\| \frac{\partial \mathbf{X}}{\partial s} \times \frac{\partial \mathbf{X}}{\partial t} \right\| ds\, dt.$$

For any continuous vector field $\mathbf{F} : \mathbf{X}(D) \to \mathbb{R}^3$, the **vector integral** of $\mathbf{F}$ along $\mathbf{X}$ is

$$\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}\left(\mathbf{X}(s,t)\right) \cdot \left( \frac{\partial \mathbf{X}}{\partial s} \times \frac{\partial \mathbf{X}}{\partial t} \right) ds\, dt.$$

Equivalently, $\displaystyle\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} = \iint_D \begin{vmatrix} F_1 & F_2 & F_3 \\ \frac{\partial X_1}{\partial s} & \frac{\partial X_2}{\partial s} & \frac{\partial X_3}{\partial s} \\ \frac{\partial X_1}{\partial t} & \frac{\partial X_2}{\partial t} & \frac{\partial X_3}{\partial t} \end{vmatrix} ds\, dt.$

# Surface integrals and reparametrization

Given two smooth parametrized surfaces $\mathbf{X} : D_1 \to \mathbb{R}^3$ and $\mathbf{Y} : D_2 \to \mathbb{R}^3$, we say that $\mathbf{Y}$ is a **smooth reparametrization** of $\mathbf{X}$ if there exists an invertible function $\mathbf{H} : D_2 \to D_1$ such that $\mathbf{Y} = \mathbf{X} \circ \mathbf{H}$ and both $\mathbf{H}$ and $\mathbf{H}^{-1}$ are smooth.

**Theorem 1** Any scalar surface integral is invariant under smooth reparametrizations.

Any smooth parametrization of a surface defines an **orientation** on it (continuous, unit normal vector field $\mathbf{n}$).

**Theorem 2** Any vector surface integral is invariant under smooth orientation-preserving reparametrizations and changes its sign under orientation-reversing reparametrizations.

As a consequence, we can define the integral of a function over a non-parametrized smooth surface and the integral of a vector field along a non-parametrized, oriented smooth surface.

## Gauss's Theorem

**Theorem** Let $D \subset \mathbb{R}^3$ be a closed, bounded region with piecewise smooth boundary $\partial D$ (not necessarily connected) oriented by **outward** unit normals to $D$. Then for any smooth vector field $\mathbf{F}$ on $D$,

$$\oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{F} \, dV.$$

**Corollary** If a smooth vector field $\mathbf{F} : D \to \mathbb{R}^3$ has no divergence, $\nabla \cdot \mathbf{F} = 0$, then $\oiint_C \mathbf{F} \cdot d\mathbf{S} = 0$ for any closed, piecewise smooth surface $C$ that bounds a subregion of $D$.

# Gauss' Theorem

*Proof in the case* $D = [0,1]\times[0,1]\times[0,1]$ *and* $\mathbf{F} = (0,0,P)$*:*

$$\int_0^1 \frac{\partial P}{\partial z}(x,y,\zeta)\,d\zeta = P(x,y,1) - P(x,y,0)$$

for any $x, y \in [0,1]$ due to the Fundamental Theorem of Calculus. Integrating this equality over the unit square $Q = [0,1]\times[0,1]$, we obtain

$$\iiint_D \frac{\partial P}{\partial z}\,dV = \iint_Q P(x,y,1)\,dx\,dy - \iint_Q P(x,y,0)\,dx\,dy.$$

The first integral in the right-hand side equals the integral of the field $\mathbf{F}$ along the top face $Q\times\{1\}$ of the cube $D$ (oriented by the upward unit normals). The second integral equals the integral of $\mathbf{F}$ along the bottom face $Q\times\{0\}$ (oriented likewise). Note that integrals of $\mathbf{F}$ along the other faces of $D$ are 0 (since $\mathbf{F}$ is parallel to those faces). It follows that the entire right-hand side equals the integral of $\mathbf{F}$ along $\partial D$.

**Problem.** Let $C$ denote the closed cylinder with bottom given by $z = 0$, top given by $z = 4$, and lateral surface given by $x^2 + y^2 = 9$. We orient $\partial C$ with outward normals. Find the integral of a vector field $\mathbf{F}(x, y, z) = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ along $\partial C$.

First let us evaluate the integral directly.

The top of the cylinder is parametrized by $\mathbf{X}_{\rm top} : D \to \mathbb{R}^3$, $\mathbf{X}_{\rm top}(x, y) = (x, y, 4)$, where

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9\}.$$

The bottom is parametrized by $\mathbf{X}_{\rm bot} : D \to \mathbb{R}^3$, $\mathbf{X}_{\rm bot}(x, y) = (x, y, 0)$.

The lateral surface is parametrized by $\mathbf{X}_{\rm lat} : [0, 2\pi] \times [0, 4] \to \mathbb{R}^3$, $\mathbf{X}_{\rm lat}(\phi, z) = (3\cos\phi, 3\sin\phi, z)$.

We have $\frac{\partial \mathbf{X}_{\mathrm{top}}}{\partial x} = (1,0,0)$, $\frac{\partial \mathbf{X}_{\mathrm{top}}}{\partial y} = (0,1,0)$. Hence $\frac{\partial \mathbf{X}_{\mathrm{top}}}{\partial x} \times \frac{\partial \mathbf{X}_{\mathrm{top}}}{\partial y} = \mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$.

Since $\mathbf{X}_{\mathrm{bot}} = \mathbf{X}_{\mathrm{top}} - (0,0,4)$, we also have $\frac{\partial \mathbf{X}_{\mathrm{bot}}}{\partial x} = \mathbf{e}_1$, $\frac{\partial \mathbf{X}_{\mathrm{bot}}}{\partial y} = \mathbf{e}_2$, and $\frac{\partial \mathbf{X}_{\mathrm{bot}}}{\partial x} \times \frac{\partial \mathbf{X}_{\mathrm{bot}}}{\partial y} = \mathbf{e}_3$.

Further, $\frac{\partial \mathbf{X}_{\mathrm{lat}}}{\partial \phi} = (-3\sin\phi, 3\cos\phi, 0)$ and $\frac{\partial \mathbf{X}_{\mathrm{lat}}}{\partial z} = (0,0,1)$. Therefore

$$\frac{\partial \mathbf{X}_{\mathrm{lat}}}{\partial \phi} \times \frac{\partial \mathbf{X}_{\mathrm{lat}}}{\partial z} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ -3\sin\phi & 3\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = (3\cos\phi, 3\sin\phi, 0).$$

We observe that $\mathbf{X}_{\mathrm{top}}$ and $\mathbf{X}_{\mathrm{lat}}$ agree with the orientation of the surface $C$ while $\mathbf{X}_{\mathrm{bot}}$ does not. It follows that

$$\oiint_{\partial C} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathbf{X}_{\mathrm{top}}} \mathbf{F} \cdot d\mathbf{S} - \iint_{\mathbf{X}_{\mathrm{bot}}} \mathbf{F} \cdot d\mathbf{S} + \iint_{\mathbf{X}_{\mathrm{lat}}} \mathbf{F} \cdot d\mathbf{S}.$$

Integrating the vector field $\mathbf{F} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ along each part of the boundary of $C$, we obtain:

$$\iint_{\mathbf{X}_{\text{top}}} \mathbf{F}\cdot d\mathbf{S} = \iint_D (x,y,4)\cdot(0,0,1)\,dx\,dy = \iint_D 4\,dx\,dy = 36\pi,$$

$$\iint_{\mathbf{X}_{\text{bot}}} \mathbf{F}\cdot d\mathbf{S} = \iint_D (x,y,0)\cdot(0,0,1)\,dx\,dy = \iint_D 0\,dx\,dy = 0,$$

$$\iint_{\mathbf{X}_{\text{lat}}} \mathbf{F}\cdot d\mathbf{S} =$$

$$= \iint_{[0,2\pi]\times[0,4]} (3\cos\phi, 3\sin\phi, z)\cdot(3\cos\phi, 3\sin\phi, 0)\,d\phi\,dz$$

$$= \iint_{[0,2\pi]\times[0,4]} 9\,d\phi\,dz = 72\pi.$$

Thus $\oiint_{\partial C} \mathbf{F}\cdot d\mathbf{S} = 36\pi - 0 + 72\pi = 108\pi.$

**Problem.** Let $C$ denote the closed cylinder with bottom given by $z = 0$, top given by $z = 4$, and lateral surface given by $x^2 + y^2 = 9$. We orient $\partial C$ with outward normals. Find the integral of a vector field $\mathbf{F}(x, y, z) = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ along $\partial C$.

Now let us use Gauss' Theorem:

$$\oiint_{\partial C} \mathbf{F} \cdot d\mathbf{S} = \iiint_C \nabla \cdot \mathbf{F} \, dV$$

$$= \iiint_C \left( \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) \right) dx\, dy\, dz$$

$$= \iiint_C 3 \, dx\, dy\, dz = 108\pi.$$

# Stokes's Theorem

Suppose $S$ is an oriented surface in $\mathbb{R}^3$ bounded by an oriented curve $\partial S$. We say that $\partial S$ is **oriented consistently with** $S$ if, as one traverses $\partial S$, the surface $S$ is on the left when looking down from the tip of $\mathbf{n}$, the unit normal vector indicating the orientation of $S$.

**Theorem** Let $S \subset \mathbb{R}^3$ be a bounded, piecewise smooth oriented surface with piecewise smooth boundary $\partial S$ oriented consistently with $S$. Then for any smooth vector field $\mathbf{F}$ on $S$,

$$\iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s}.$$

**Corollary** If the surface $S$ is closed (i.e., has no boundary), then for any smooth vector field $\mathbf{F}$ on $S$,

$$\iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = 0.$$

# Example

Suppose that a bounded, piecewise smooth surface $S \subset \mathbb{R}^3$ is contained in the *xy*-coordinate plane, that is, $S = D \times \{0\}$ for a domain $D \subset \mathbb{R}^2$. We orient $S$ by the upward unit normal vector $\mathbf{n} = (0, 0, 1)$ and orient the boundary $\partial S = \partial D \times \{0\}$ consistently with $S$. Further, suppose that $\mathbf{F}$ is a horizontal vector field, $\mathbf{F} = (M, N, 0)$. By Stokes' Theorem,

$$\iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s}.$$

Recall that $\iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = \iint_S \operatorname{curl}(\mathbf{F}) \cdot \mathbf{n}\, dS$. We obtain

$$\operatorname{curl}(\mathbf{F}) \cdot \mathbf{n} = \begin{vmatrix} 0 & 0 & 1 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.$$

It follows that this particular case of Stokes' Theorem is equivalent to Green's Theorem.