MATH 311

Topics in Applied Mathematics I

Lecture 37:
Review for the final exam.

Topics for the final exam: Part I

Elementary linear algebra (L/C 1.1–1.5, 2.1–2.2)

- Systems of linear equations: elementary operations, Gaussian elimination, back substitution.
- Matrix of coefficients and augmented matrix. Elementary row operations, row echelon form and reduced row echelon form.
- Matrix algebra. Inverse matrix.
- Determinants: explicit formulas for 2×2 and 3×3 matrices, row and column expansions, elementary row and column operations.

Topics for the final exam: Part II

Abstract linear algebra (L/C 3.1–3.6, 4.1–4.3)

- Vector spaces (vectors, matrices, polynomials, functional spaces).
- Subspaces. Nullspace, column space, and row space of a matrix.
- Span, spanning set. Linear independence.
- Bases and dimension.
- Rank and nullity of a matrix.
- Coordinates relative to a basis.
- Change of basis, transition matrix.

- Linear transformations.
- Matrix transformations.
- Matrix of a linear mapping.
- Change of basis for a linear operator.
- Similarity of matrices.

Topics for the final exam: Part III

Advanced linear algebra (L/C 5.1–5.6, 6.1, 6.3)

- Eigenvalues, eigenvectors, eigenspaces
- Characteristic polynomial
- Bases of eigenvectors, diagonalization

- Euclidean structure in $\mathbb{R}^n$ (length, angle, dot product)
- Inner products and norms
- Orthogonal complement, orthogonal projection
- Least squares problems
- The Gram-Schmidt orthogonalization process

Topics for the final exam: Part IV

Vector analysis (L/C 8.1–8.4, 9.1–9.5, 10.1–10.3, 11.1–11.3)

- Gradient, divergence, and curl
- Fubini's Theorem
- Change of coordinates in a multiple integral
- Geometric meaning of the determinant
- Length of a curve
- Line integrals
- Green's Theorem
- Conservative vector fields
- Area of a surface
- Surface integrals
- Gauss' Theorem
- Stokes' Theorem

Problem. Let V be the vector space spanned by functions $f_1(x) = x \sin x$, $f_2(x) = x \cos x$, $f_3(x) = \sin x$, and $f_4(x) = \cos x$.
Consider the linear operator $D : V \to V$, $D = d/dx$.

(a) Find the matrix A of the operator D relative to the basis f_1, f_2, f_3, f_4.
(b) Find the eigenvalues of A.
(c) Is the matrix A diagonalizable?

A is a 4×4 matrix whose columns are coordinates of functions $Df_i = f_i'$ relative to the basis f_1, f_2, f_3, f_4.

$$f_1'(x) = (x \sin x)' = x \cos x + \sin x = f_2(x) + f_3(x),$$
$$\begin{aligned} f_2'(x) &= (x \cos x)' = -x \sin x + \cos x \\ &= -f_1(x) + f_4(x), \end{aligned}$$
$$f_3'(x) = (\sin x)' = \cos x = f_4(x),$$
$$f_4'(x) = (\cos x)' = -\sin x = -f_3(x).$$

Thus $A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$

Eigenvalues of A are roots of its characteristic polynomial

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & -1 & 0 & 0 \\ 1 & -\lambda & 0 & 0 \\ 1 & 0 & -\lambda & -1 \\ 0 & 1 & 1 & -\lambda \end{vmatrix}$$

Expand the determinant by the 1st row:

$$\det(A - \lambda I) = -\lambda \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & -1 \\ 1 & 1 & -\lambda \end{vmatrix} - (-1) \begin{vmatrix} 1 & 0 & 0 \\ 1 & -\lambda & -1 \\ 0 & 1 & -\lambda \end{vmatrix}$$

$$= \lambda^2(\lambda^2+1) + (\lambda^2+1) = (\lambda^2+1)^2 = (\lambda-i)^2(\lambda+i)^2.$$

The roots are i and $-i$, both of multiplicity 2.

One can show that both eigenspaces of A are one-dimensional. The eigenspace for i is spanned by $(0, 0, i, 1)$ and the eigenspace for $-i$ is spanned by $(0, 0, -i, 1)$. It follows that the matrix A is not diagonalizable in the complex vector space $\mathbb{C}^4$ (let alone real vector space $\mathbb{R}^4$).

There is also an indirect way to show that A is not diagonalizable. Assume the contrary. Then $A = UPU^{-1}$, where U is an invertible matrix with complex entries and

$$P = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$$

(note that P should have the same characteristic polynomial as A). This would imply that $A^2 = UP^2U^{-1}$. But $P^2 = -I$ so that $A^2 = U(-I)U^{-1} = -I$.

Let us check if $A^2 = -I$.

$$A^2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -2 & -1 & 0 \\ 2 & 0 & 0 & -1 \end{pmatrix}.$$

Since $A^2 \neq -I$, we have a contradiction. Thus the matrix A is not diagonalizable in $\mathbb{C}^4$.

Problem. Consider a linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v}$, where $\mathbf{v}_0 = (3/5, 0, -4/5)$.

(a) Find the matrix B of the operator L.
(b) Find the range and kernel of L.
(c) Find the eigenvalues of L.
(d) Find the matrix of the operator L^{2017} (L applied 2017 times).

$$L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v}, \quad \mathbf{v}_0 = (3/5, 0, -4/5).$$

Let $\mathbf{v} = (x, y, z) = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$. Then

$$L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 3/5 & 0 & -4/5 \\ x & y & z \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -4/5 \\ y & z \end{vmatrix} \mathbf{e}_1 - \begin{vmatrix} 3/5 & -4/5 \\ x & z \end{vmatrix} \mathbf{e}_2 + \begin{vmatrix} 3/5 & 0 \\ x & y \end{vmatrix} \mathbf{e}_3$$

$$= \tfrac{4}{5}y\mathbf{e}_1 - \left(\tfrac{4}{5}x + \tfrac{3}{5}z\right)\mathbf{e}_2 + \tfrac{3}{5}y\mathbf{e}_3 = \left(\tfrac{4}{5}y, -\tfrac{4}{5}x - \tfrac{3}{5}z, \tfrac{3}{5}y\right).$$

In particular, $L(\mathbf{e}_1) = \left(0, -\frac{4}{5}, 0\right)$, $L(\mathbf{e}_2) = \left(\frac{4}{5}, 0, \frac{3}{5}\right)$, $L(\mathbf{e}_3) = \left(0, -\frac{3}{5}, 0\right)$.

Therefore $B = \begin{pmatrix} 0 & 4/5 & 0 \\ -4/5 & 0 & -3/5 \\ 0 & 3/5 & 0 \end{pmatrix}$.

The range of the operator L is spanned by columns of the matrix B. It follows that $\mathrm{Range}(L)$ is the plane spanned by $\mathbf{v}_1 = (0,1,0)$ and $\mathbf{v}_2 = (4,0,3)$.

The kernel of L is the nullspace of the matrix B, i.e., the solution set for the equation $B\mathbf{x} = \mathbf{0}$.

$$\begin{pmatrix} 0 & 4/5 & 0 \\ -4/5 & 0 & -3/5 \\ 0 & 3/5 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3/4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Longrightarrow\ x + \tfrac{3}{4}z = y = 0 \ \Longrightarrow\ \mathbf{x} = t(-3/4, 0, 1).$$

Alternatively, the kernel of L is the set of vectors $\mathbf{v} \in \mathbb{R}^3$ such that $L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v} = \mathbf{0}$.
It follows that this is the line spanned by $\mathbf{v}_0 = (3/5, 0, -4/5)$.

Characteristic polynomial of the matrix B:

$$\det(B - \lambda I) = \begin{vmatrix} -\lambda & 4/5 & 0 \\ -4/5 & -\lambda & -3/5 \\ 0 & 3/5 & -\lambda \end{vmatrix}$$

$$= -\lambda^3 - (3/5)^2\lambda - (4/5)^2\lambda = -\lambda^3 - \lambda = -\lambda(\lambda^2 + 1).$$

The eigenvalues are 0, i, and $-i$.

The matrix of the operator L^{2017} is B^{2017}.

Since the matrix B has eigenvalues 0, i, and $-i$, it is diagonalizable in $\mathbb{C}^3$. Namely, $B = UDU^{-1}$, where U is an invertible matrix with complex entries and

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}.$$

Then $B^{2017} = UD^{2017}U^{-1}$. We have that $D^{2017} =$
$= \mathrm{diag}\big(0, i^{2017}, (-i)^{2017}\big) = \mathrm{diag}(0, i, -i) = D$.
Hence

$$B^{2017} = UDU^{-1} = B = \begin{pmatrix} 0 & 4/5 & 0 \\ -4/5 & 0 & -3/5 \\ 0 & 3/5 & 0 \end{pmatrix}.$$

Problem. Find a quadratic polynomial that is the best least squares fit to the function $f(x) = |x|$ on the interval $[-1, 1]$.

The best least squares fit is a polynomial $q(x)$ that minimizes the distance relative to the integral norm

$$\|f - q\| = \left(\int_{-1}^{1} |f(x) - q(x)|^2 \, dx \right)^{1/2}$$

over all polynomials of degree 2.

The norm $\|\cdot\|$ is induced by the inner product

$$\langle g, h\rangle = \int_{-1}^{1} g(x)h(x)\,dx.$$

Therefore $\|f - p\|$ is minimal if p is the orthogonal projection of the function f on the subspace $\mathcal{P}_3$ of quadratic polynomials.

Suppose that p_0, p_1, p_2 is an orthogonal basis for $\mathcal{P}_3$. Then

$$q(x) = \frac{\langle f, p_0\rangle}{\langle p_0, p_0\rangle} p_0(x) + \frac{\langle f, p_1\rangle}{\langle p_1, p_1\rangle} p_1(x) + \frac{\langle f, p_2\rangle}{\langle p_2, p_2\rangle} p_2(x).$$

An orthogonal basis can be obtained by applying the *Gram-Schmidt orthogonalization process* to the basis $1, x, x^2$:

$$p_0(x) = 1,$$

$$p_1(x) = x - \frac{\langle x, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0(x) = x - \frac{\langle x, 1 \rangle}{\langle 1, 1 \rangle} = x,$$

$$\begin{aligned} p_2(x) &= x^2 - \frac{\langle x^2, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0(x) - \frac{\langle x^2, p_1 \rangle}{\langle p_1, p_1 \rangle} p_1(x) \\ &= x^2 - \frac{\langle x^2, 1 \rangle}{\langle 1, 1 \rangle} - \frac{\langle x^2, x \rangle}{\langle x, x \rangle} x = x^2 - \frac{1}{3}. \end{aligned}$$

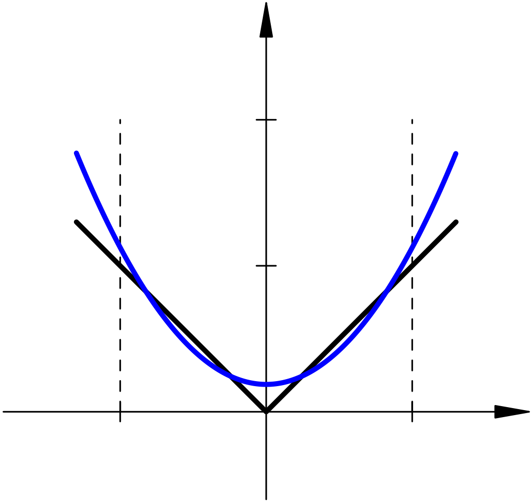
Problem. Find a quadratic polynomial that is the best least squares fit to the function $f(x) = |x|$ on the interval $[-1, 1]$.

Solution:

$$q(x) = \frac{\langle f, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0(x) + \frac{\langle f, p_1 \rangle}{\langle p_1, p_1 \rangle} p_1(x) + \frac{\langle f, p_2 \rangle}{\langle p_2, p_2 \rangle} p_2(x)$$

$$= \frac{1}{2} p_0(x) + \frac{15}{16} p_2(x)$$

$$= \frac{1}{2} + \frac{15}{16}\left(x^2 - \frac{1}{3}\right) = \frac{3}{16}(5x^2 + 1).$$



Area, volume, and determinants

- 2×2 determinants and plane geometry

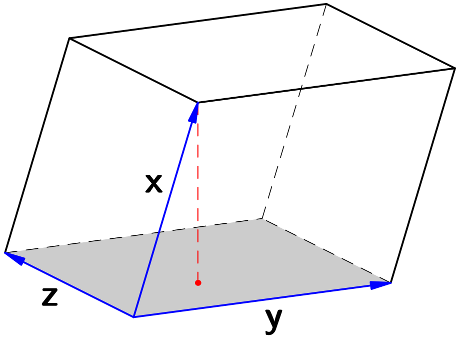
Let P be a parallelogram in the plane $\mathbb{R}^2$. Suppose that vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$ are represented by adjacent sides of P. Then $\mathrm{area}(P) = |\det A|$, where $A = (\mathbf{v}_1, \mathbf{v}_2)$, a matrix whose columns are $\mathbf{v}_1$ and $\mathbf{v}_2$.

Consider a linear operator $L_A : \mathbb{R}^2 \to \mathbb{R}^2$ given by $L_A(\mathbf{v}) = A\mathbf{v}$ for any column vector $\mathbf{v}$. Then $\mathrm{area}(L_A(D)) = |\det A| \,\mathrm{area}(D)$ for any bounded domain D.

- 3×3 determinants and space geometry

Let Π be a parallelepiped in space $\mathbb{R}^3$. Suppose that vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$ are represented by adjacent edges of Π. Then $\mathrm{volume}(\Pi) = |\det B|$, where $B = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, a matrix whose columns are $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

Similarly, $\mathrm{volume}(L_B(D)) = |\det B| \,\mathrm{volume}(D)$ for any bounded domain $D \subset \mathbb{R}^3$.

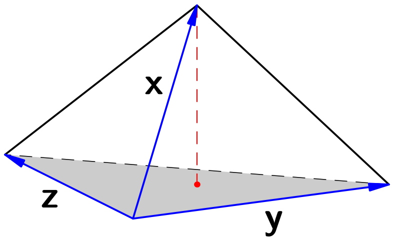


Parallelepiped is a prism.

(Volume) $=$ (area of the base) $\times$ (height)

Area of the base $= |\mathbf{y} \times \mathbf{z}|$

Volume $= |\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z})|$



Tetrahedron is a pyramid.

$(\text{Volume}) = \frac{1}{3}(\text{area of the base}) \times (\text{height})$

Area of the base $= \frac{1}{2}|\mathbf{y} \times \mathbf{z}|$

$\implies$ Volume $= \frac{1}{6}|\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z})|$