

MATH 311

Topics in Applied Mathematics I

Lecture 17:

Rank and nullity of a matrix.

Basis and coordinates.

Row space of a matrix

Definition. The **row space** of an $m \times n$ matrix A is the subspace of \mathbb{R}^n spanned by rows of A .

The dimension of the row space is called the **rank** of the matrix A .

Theorem 1 The rank of a matrix A is the maximal number of linearly independent rows in A .

Theorem 2 Elementary row operations do not change the row space of a matrix.

Theorem 3 If a matrix A is in row echelon form, then the nonzero rows of A are linearly independent.

Corollary The rank of a matrix is equal to the number of nonzero rows in its row echelon form.

Column space of a matrix

Definition. The **column space** of an $m \times n$ matrix A is the subspace of \mathbb{R}^m spanned by columns of A .

Theorem 1 The column space of a matrix A coincides with the row space of the transpose matrix A^T .

Theorem 2 Elementary row operations do not change linear relations between columns of a matrix.

Theorem 3 Elementary row operations do not change the dimension of the column space of a matrix (however they can change the column space).

Theorem 4 If a matrix is in row echelon form, then the columns with leading entries form a basis for the column space.

Corollary For any matrix, the row space and the column space have the same dimension.

Problem. Find a basis for the column space of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

The column space of A coincides with the row space of A^T .
To find a basis, we convert A^T to row echelon form:

$$A^T = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Vectors $(1, 0, 2, 1)$, $(0, 1, 1, 0)$, and $(0, 0, 0, 1)$ form a basis for the column space of A .

Problem. Find a basis for the column space of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Alternative solution: We already know from the previous lecture that the rank of A is 3. It follows that the columns of A are linearly independent. Therefore these columns form a basis for the column space.

Problem. Let V be a vector space spanned by vectors $\mathbf{w}_1 = (1, 1, 0)$, $\mathbf{w}_2 = (0, 1, 1)$, $\mathbf{w}_3 = (2, 3, 1)$, and $\mathbf{w}_4 = (1, 1, 1)$. Pare this spanning set to a basis for V .

Alternative solution: The vector space V is the column space of a matrix

$$B = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

The row echelon form of B is $C = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Columns of C with leading entries (1st, 2nd, and 4th) form a basis for the column space of C . It follows that the corresponding columns of B (i.e., 1st, 2nd, and 4th) form a basis for the column space of B .

Thus $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4\}$ is a basis for V .

Nullspace of a matrix

Let $A = (a_{ij})$ be an $m \times n$ matrix.

Definition. The **nullspace** of the matrix A , denoted $N(A)$, is the set of all n -dimensional column vectors \mathbf{x} such that

$$\boxed{A\mathbf{x} = \mathbf{0}.}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The nullspace $N(A)$ is the solution set of a system of linear homogeneous equations (with A as the coefficient matrix).

Theorem $N(A)$ is a subspace of the vector space \mathbb{R}^n .

Definition. The dimension of the nullspace $N(A)$ is called the **nullity** of the matrix A .

rank + nullity

Theorem The rank of a matrix A plus the nullity of A equals the number of columns in A .

Sketch of the proof: The rank of A equals the number of nonzero rows in the row echelon form, which equals the number of leading entries.

The nullity of A equals the number of free variables in the corresponding homogeneous system, which equals the number of columns without leading entries in the row echelon form.

Consequently, rank+nullity is the number of all columns in the matrix A .

Problem. Find the nullity of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \end{pmatrix}.$$

Clearly, the rows of A are linearly independent.

Therefore the rank of A is 2. Since

$$(\text{rank of } A) + (\text{nullity of } A) = 4,$$

it follows that the nullity of A is 2.

Basis and dimension

Definition. Let V be a vector space. A linearly independent spanning set for V is called a **basis**.

Theorem Any vector space V has a basis. If V has a finite basis, then all bases for V are finite and have the same number of elements (called the *dimension* of V).

Example. Vectors $\mathbf{e}_1 = (1, 0, 0, \dots, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0, \dots, 0, 0), \dots, \mathbf{e}_n = (0, 0, 0, \dots, 0, 1)$ form a basis for \mathbb{R}^n (called *standard*) since

$$(x_1, x_2, \dots, x_n) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n.$$

Basis and coordinates

If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for a vector space V , then any vector $\mathbf{v} \in V$ has a unique representation

$$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \dots + x_n\mathbf{v}_n,$$

where $x_i \in \mathbb{R}$. The coefficients x_1, x_2, \dots, x_n are called the **coordinates** of \mathbf{v} with respect to the ordered basis $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$.

The mapping

$$\text{vector } \mathbf{v} \mapsto \text{its coordinates } (x_1, x_2, \dots, x_n)$$

is a one-to-one correspondence between V and \mathbb{R}^n . This correspondence respects linear operations in V and in \mathbb{R}^n .

Examples. • Coordinates of a vector

$\mathbf{v} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ relative to the standard basis $\mathbf{e}_1 = (1, 0, \dots, 0, 0)$, $\mathbf{e}_2 = (0, 1, \dots, 0, 0), \dots$, $\mathbf{e}_n = (0, 0, \dots, 0, 1)$ are (x_1, x_2, \dots, x_n) .

• Coordinates of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_{2,2}(\mathbb{R})$

relative to the basis $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$
 $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are (a, c, b, d) .

• Coordinates of a polynomial

$p(x) = a_0 + a_1x + \dots + a_{n-1}x^{n-1} \in \mathcal{P}_n$ relative to the basis $1, x, x^2, \dots, x^{n-1}$ are $(a_0, a_1, \dots, a_{n-1})$.

Weird vector space

Consider the set $V = \mathbb{R}_+$ of positive numbers with a nonstandard addition and scalar multiplication:

$$\boxed{x \oplus y = xy} \quad \text{for any } x, y \in \mathbb{R}_+.$$

$$\boxed{r \odot x = x^r} \quad \text{for any } x \in \mathbb{R}_+ \text{ and } r \in \mathbb{R}.$$

This is an example of a vector space.

The zero vector in V is the number 1. To build a basis for V , we can begin with any number $v \in V$ different from 1. Let's take $v = 2$. The span $\text{Span}(2)$ consists of all numbers of the form $r \odot 2 = 2^r$, $r \in \mathbb{R}$. It is the entire space V . Hence $\{2\}$ is a basis for V so that $\dim V = 1$.

The coordinate mapping $f : V \rightarrow \mathbb{R}$ associated to this basis is given by $f(2^r) = r$ for all $r \in \mathbb{R}$. Equivalently, $f(x) = \log_2 x$, $x \in V$. Notice that $\log_2(x \oplus y) = \log_2 x + \log_2 y$ and $\log_2(r \odot x) = r \log_2 x$.

Vectors $\mathbf{u}_1=(3, 1)$ and $\mathbf{u}_2=(2, 1)$ form a basis for \mathbb{R}^2 .

Problem 1. Find coordinates of the vector $\mathbf{v} = (7, 4)$ with respect to the basis $\mathbf{u}_1, \mathbf{u}_2$.

The desired coordinates x, y satisfy

$$\mathbf{v} = x\mathbf{u}_1 + y\mathbf{u}_2 \iff \begin{cases} 3x + 2y = 7 \\ x + y = 4 \end{cases} \iff \begin{cases} x = -1 \\ y = 5 \end{cases}$$

Problem 2. Find the vector \mathbf{w} whose coordinates with respect to the basis $\mathbf{u}_1, \mathbf{u}_2$ are $(7, 4)$.

$$\mathbf{w} = 7\mathbf{u}_1 + 4\mathbf{u}_2 = 7(3, 1) + 4(2, 1) = (29, 11)$$

Change of coordinates

Given a vector $\mathbf{v} \in \mathbb{R}^2$, let (x, y) be its standard coordinates, i.e., coordinates with respect to the standard basis $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1)$, and let (x', y') be its coordinates with respect to the basis $\mathbf{u}_1 = (3, 1)$, $\mathbf{u}_2 = (2, 1)$.

Problem. Find a relation between (x, y) and (x', y') .

By definition, $\mathbf{v} = x\mathbf{e}_1 + y\mathbf{e}_2 = x'\mathbf{u}_1 + y'\mathbf{u}_2$.

In standard coordinates,

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= x' \begin{pmatrix} 3 \\ 1 \end{pmatrix} + y' \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \\ \implies \begin{pmatrix} x' \\ y' \end{pmatrix} &= \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

Change of coordinates in \mathbb{R}^n

The usual (standard) coordinates of a vector $\mathbf{v} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ are coordinates relative to the standard basis $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$. Let $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ be another basis for \mathbb{R}^n and $(x'_1, x'_2, \dots, x'_n)$ be the coordinates of the same vector \mathbf{v} with respect to this basis. Then

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ u_{21} & u_{22} & \dots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & \dots & u_{nn} \end{pmatrix} \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix},$$

where the matrix $U = (u_{ij})$ does not depend on the vector \mathbf{v} . Namely, columns of U are coordinates of vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ with respect to the standard basis. U is called the **transition matrix** from the basis $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ to the standard basis $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$. The inverse matrix U^{-1} is called the **transition matrix** from $\mathbf{e}_1, \dots, \mathbf{e}_n$ to $\mathbf{u}_1, \dots, \mathbf{u}_n$.