MATH 311

Topics in Applied Mathematics I

Lecture 21a:
Similar matrices.

Change of basis for a linear operator

Let $L : V \to V$ be a linear operator on a vector space V.

Let A be the matrix of L relative to a basis $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ for V. Let B be the matrix of L relative to another basis $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n$ for V.

Let U be the transition matrix from the basis $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ to $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n$.

$$
\begin{array}{ccc}
\boxed{\mathbf{a}\text{-coordinates of } \mathbf{v}} & \xrightarrow{A} & \boxed{\mathbf{a}\text{-coordinates of } L(\mathbf{v})} \\
U \big\downarrow & & \big\downarrow U \\
\boxed{\mathbf{b}\text{-coordinates of } \mathbf{v}} & \xrightarrow{B} & \boxed{\mathbf{b}\text{-coordinates of } L(\mathbf{v})}
\end{array}
$$

It follows that $UA\mathbf{x} = BU\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n \implies UA = BU$.
Then $A = U^{-1}BU$ and $B = UAU^{-1}$.

Similarity of matrices

Definition. An $n \times n$ matrix B is said to be **similar** to an $n \times n$ matrix A if $\boxed{B = S^{-1}AS}$ for some nonsingular $n \times n$ matrix S.

Remark. Two $n \times n$ matrices are similar if and only if they represent the same linear operator on $\mathbb{R}^n$ with respect to different bases.

Theorem Similarity is an *equivalence relation*, which means that **(i)** any square matrix A is similar to itself;
(ii) if B is similar to A, then A is similar to B;
(iii) if A is similar to B and B is similar to C, then A is similar to C.

Corollary The set of $n \times n$ matrices is partitioned into disjoint subsets (called *similarity classes*) such that all matrices in the same subset are similar to each other while matrices from different subsets are never similar.

Theorem Similarity is an *equivalence relation*, i.e.,
(i) any square matrix A is similar to itself;
(ii) if B is similar to A, then A is similar to B;
(iii) if A is similar to B and B is similar to C, then A is similar to C.

Proof: **(i)** $A = I^{-1}AI$.

(ii) If $B = S^{-1}AS$ then $A = SBS^{-1} = (S^{-1})^{-1}BS^{-1}$
$= S_1^{-1}BS_1,$ where $S_1 = S^{-1}$.

(iii) If $A = S^{-1}BS$ and $B = T^{-1}CT$ then
$A = S^{-1}(T^{-1}CT)S = (S^{-1}T^{-1})C(TS) = (TS)^{-1}C(TS)$
$= S_2^{-1}CS_2,$ where $S_2 = TS$.

Theorem If A and B are similar matrices then they have the same **(i)** determinant, **(ii)** trace = the sum of diagonal entries, **(iii)** rank, and **(iv)** nullity.