MATH 311

Topics in Applied Mathematics I

# Lecture 27:
# Review for Test 2.

## Topics for Test 2

*Vector spaces (Leon/Colley 3.4–3.6)*

- Basis and dimension
- Rank and nullity of a matrix
- Coordinates relative to a basis
- Change of basis, transition matrix

*Linear transformations (Leon/Colley 4.1–4.3)*

- Linear transformations
- Range and kernel
- Matrix transformations
- Matrix of a linear transformation
- Change of basis for a linear operator
- Similar matrices

## Topics for Test 2

*Eigenvalues and eigenvectors (Leon/Colley 6.1, 6.3)*

- Eigenvalues, eigenvectors, eigenspaces
- Characteristic polynomial
- Diagonalization

*Orthogonality (Leon/Colley 5.1–5.3, 5.5–5.6)*

- Orthogonal complement
- Orthogonal projection
- Least squares problems
- The Gram-Schmidt orthogonalization process

# Sample problems for Test 2

**Problem 1** Let $A = \begin{pmatrix} 0 & -1 & 4 & 1 \\ 1 & 1 & 2 & -1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}$.

**(i)** Find the rank and the nullity of the matrix $A$.
**(ii)** Find a basis for the row space of $A$, then extend this basis to a basis for $\mathbb{R}^4$.
**(iii)** Find a basis for the nullspace of $A$.

**Problem 2** Let $A$ and $B$ be two matrices such that the product $AB$ is well defined.

(i) Prove that $\text{rank}(AB) \le \text{rank}(B)$.
(ii) Prove that $\text{rank}(AB) \le \text{rank}(A)$.

# Sample problems for Test 2

**Problem 3** Let $V$ be a subspace of $F(\mathbb{R})$ spanned by functions $e^x$ and $e^{-x}$. Let $L$ be a linear operator on $V$ such that $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ is the matrix of $L$ relative to the basis $e^x$, $e^{-x}$. Find the matrix of $L$ relative to the basis $\cosh x = \frac{1}{2}(e^x + e^{-x})$, $\sinh x = \frac{1}{2}(e^x - e^{-x})$.

**Problem 4** Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

**(i)** Find all eigenvalues of the matrix $A$.
**(ii)** For each eigenvalue of $A$, find an associated eigenvector.
**(iii)** Is the matrix $A$ diagonalizable? Explain.
**(iv)** Find all eigenvalues of the matrix $A^2$.

# Sample problems for Test 2

**Problem 5** Find a linear polynomial which is the best least squares fit to the following data:

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $f(x)$ | $-3$ | $-2$ | $1$ | $2$ | $5$ |

**Problem 6** Let $V$ be a subspace of $\mathbb{R}^4$ spanned by the vectors $\mathbf{x}_1 = (1,1,1,1)$ and $\mathbf{x}_2 = (1,0,3,0)$.

**(i)** Find an orthonormal basis for $V$.

**(ii)** Find an orthonormal basis for the orthogonal complement $V^\perp$.

**(iii)** Find the distance from the vector $\mathbf{y} = (1,0,0,0)$ to the subspaces $V$ and $V^\perp$.

**Problem 1.** Let $A = \begin{pmatrix} 0 & -1 & 4 & 1 \\ 1 & 1 & 2 & -1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}$.

**(i)** Find the rank and the nullity of the matrix $A$.

The rank (= dimension of the row space) and the nullity (= dimension of the nullspace) of a matrix are preserved under elementary row operations. We apply such operations to convert the matrix $A$ into row echelon form.

Interchange the 1st row with the 2nd row:

$$\to \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & 4 & 1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}$$

Add 3 times the 1st row to the 3rd row, then subtract 2 times the 1st row from the 4th row:

$$\rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & 4 & 1 \\ 0 & 3 & 5 & -3 \\ 2 & -1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & 4 & 1 \\ 0 & 3 & 5 & -3 \\ 0 & -3 & -4 & 3 \end{pmatrix}$$

Multiply the 2nd row by $-1$:

$$\rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 3 & 5 & -3 \\ 0 & -3 & -4 & 3 \end{pmatrix}$$

Add the 4th row to the 3rd row:

$$\rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & -4 & 3 \end{pmatrix}$$

Add 3 times the 2nd row to the 4th row:

$$\rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -16 & 0 \end{pmatrix}$$

Add 16 times the 3rd row to the 4th row:

$$\rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now that the matrix is in row echelon form, its rank equals the number of nonzero rows, which is 3. Since

$$(\text{rank of } A) + (\text{nullity of } A) = (\text{the number of columns of } A) = 4,$$

it follows that the nullity of $A$ equals 1.

**Problem 1.** Let $A = \begin{pmatrix} 0 & -1 & 4 & 1 \\ 1 & 1 & 2 & -1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}$.

**(ii)** Find a basis for the row space of $A$, then extend this basis to a basis for $\mathbb{R}^4$.

The row space of a matrix is invariant under elementary row operations. Therefore the row space of the matrix $A$ is the same as the row space of its row echelon form:

$$\begin{pmatrix} 0 & -1 & 4 & 1 \\ 1 & 1 & 2 & -1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The nonzero rows of the latter matrix are linearly independent so that they form a basis for its row space:

$$\mathbf{v}_1 = (1, 1, 2, -1), \;\; \mathbf{v}_2 = (0, 1, -4, -1), \;\; \mathbf{v}_3 = (0, 0, 1, 0).$$

To extend the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to a basis for $\mathbb{R}^4$, we need a vector $\mathbf{v}_4 \in \mathbb{R}^4$ that is not a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

It is known that at least one of the vectors $\mathbf{e}_1 = (1, 0, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0, 0)$, $\mathbf{e}_3 = (0, 0, 1, 0)$, and $\mathbf{e}_4 = (0, 0, 0, 1)$ can be chosen as $\mathbf{v}_4$.

In particular, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{e}_4$ form a basis for $\mathbb{R}^4$. This follows from the fact that the $4 \times 4$ matrix whose rows are these vectors is not singular:

$$\begin{vmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \neq 0.$$

**Problem 1.** Let $A = \begin{pmatrix} 0 & -1 & 4 & 1 \\ 1 & 1 & 2 & -1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}$.

**(iii)** Find a basis for the nullspace of $A$.

The nullspace of $A$ is the solution set of the system of linear homogeneous equations with $A$ as the coefficient matrix. To solve the system, we convert $A$ to reduced row echelon form:

$$\to \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\implies \quad x_1 = x_2 - x_4 = x_3 = 0$

General solution: $(x_1, x_2, x_3, x_4) = (0, t, 0, t) = t(0, 1, 0, 1)$.

Thus the vector $(0, 1, 0, 1)$ forms a basis for the nullspace of $A$.

**Problem 2.** Let $A$ and $B$ be two matrices such that the product $AB$ is well defined.

**(i)** Prove that $\mathrm{rank}(AB) \leq \mathrm{rank}(B)$.

Suppose that $B\mathbf{x} = \mathbf{0}$ for some column vector $\mathbf{x}$. Then $(AB)\mathbf{x} = A(B\mathbf{x}) = A\mathbf{0} = \mathbf{0}$. It follows that the nullspace of $B$ is contained in the nullspace of $AB$. Consequently, $\mathrm{nullity}(B) \leq \mathrm{nullity}(AB)$. Since matrices $AB$ and $B$ have the same number of columns, we obtain $\mathrm{rank}(AB) \leq \mathrm{rank}(B)$.

**(ii)** Prove that $\mathrm{rank}(AB) \leq \mathrm{rank}(A)$.

Note that $\mathrm{rank}(M) = \mathrm{rank}(M^T)$ for any matrix $M$.
In particular, $\mathrm{rank}(AB) = \mathrm{rank}((AB)^T) = \mathrm{rank}(B^T A^T)$.
By the above, $\mathrm{rank}(B^T A^T) \leq \mathrm{rank}(A^T) = \mathrm{rank}(A)$.

*Remark.* Alternatively, one can show that the row space of $AB$ is contained in the row space of $B$ while the column space of $AB$ is contained in the column space of $A$.

**Problem 3.** Let $V$ be a subspace of $F(\mathbb{R})$ spanned by functions $e^x$ and $e^{-x}$. Let $L$ be a linear operator on $V$ such that $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ is the matrix of $L$ relative to the basis $e^x$, $e^{-x}$. Find the matrix of $L$ relative to the basis $\cosh x = \frac{1}{2}(e^x + e^{-x})$, $\sinh x = \frac{1}{2}(e^x - e^{-x})$.

Let $A$ denote the matrix of the operator $L$ relative to the basis $e^x$, $e^{-x}$ (which is given) and $B$ denote the matrix of $L$ relative to the basis $\cosh x$, $\sinh x$ (which is to be found). By definition of the functions $\cosh x$ and $\sinh x$, the transition matrix from $\cosh x$, $\sinh x$ to $e^x$, $e^{-x}$ is $U = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. It follows that $B = U^{-1}AU$. We obtain that

$$B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \cdot \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 4 \end{pmatrix}.$$

**Problem 4.** Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

**(i)** Find all eigenvalues of the matrix $A$.

The eigenvalues of $A$ are roots of the characteristic equation $\det(A - \lambda I) = 0$. We obtain that

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 2 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)^3 - 2(1-\lambda) - 2(1-\lambda) = (1-\lambda)\big((1-\lambda)^2 - 4\big)$$

$$= (1-\lambda)\big((1-\lambda) - 2\big)\big((1-\lambda) + 2\big) = -(\lambda - 1)(\lambda + 1)(\lambda - 3).$$

Hence the matrix $A$ has three eigenvalues: $-1$, 1, and 3.

**Problem 4.** Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

**(ii)** For each eigenvalue of $A$, find an associated eigenvector.

An eigenvector $\mathbf{v} = (x, y, z)$ of the matrix $A$ associated with an eigenvalue $\lambda$ is a nonzero solution of the vector equation

$$(A - \lambda I)\mathbf{v} = \mathbf{0} \iff \begin{pmatrix} 1-\lambda & 2 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 2 & 1-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

To solve the equation, we convert the matrix $A - \lambda I$ to reduced row echelon form.

First consider the case $\lambda = -1$. The row reduction yields

$$A + I = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} \color{red}{1} & 0 & -1 \\ 0 & \color{red}{1} & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$(A + I)\mathbf{v} = \mathbf{0} \iff \begin{cases} x - z = 0, \\ y + z = 0. \end{cases}$$

The general solution is $x = t$, $y = -t$, $z = t$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_1 = (1, -1, 1)$ is an eigenvector of $A$ associated with the eigenvalue $-1$.

Secondly, consider the case $\lambda = 1$. The row reduction yields

$$A - I = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$(A - I)\mathbf{v} = \mathbf{0} \quad \Longleftrightarrow \quad \begin{cases} x + z = 0, \\ y = 0. \end{cases}$$

The general solution is $x = -t$, $y = 0$, $z = t$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_2 = (-1, 0, 1)$ is an eigenvector of $A$ associated with the eigenvalue 1.

Finally, consider the case $\lambda = 3$. The row reduction yields

$$A - 3I = \begin{pmatrix} -2 & 2 & 0 \\ 1 & -2 & 1 \\ 0 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 1 & -2 & 1 \\ 0 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & -2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$(A - 3I)\mathbf{v} = \mathbf{0} \iff \begin{cases} x - z = 0, \\ y - z = 0. \end{cases}$$

The general solution is $x = t$, $y = t$, $z = t$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_3 = (1, 1, 1)$ is an eigenvector of $A$ associated with the eigenvalue 3.

**Problem 4.** Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

**(iii)** Is the matrix $A$ diagonalizable? Explain.

The matrix $A$ is diagonalizable, i.e., there exists a basis for $\mathbb{R}^3$ formed by its eigenvectors.

Namely, the vectors $\mathbf{v}_1 = (1, -1, 1)$, $\mathbf{v}_2 = (-1, 0, 1)$, and $\mathbf{v}_3 = (1, 1, 1)$ are eigenvectors of the matrix $A$ belonging to distinct eigenvalues. Therefore these vectors are linearly independent. It follows that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is a basis for $\mathbb{R}^3$.

Alternatively, the existence of a basis for $\mathbb{R}^3$ consisting of eigenvectors of $A$ already follows from the fact that the matrix $A$ has three distinct eigenvalues.

**Problem 4.** Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

**(iv)** Find all eigenvalues of the matrix $A^2$.

Suppose that $\mathbf{v}$ is an eigenvector of the matrix $A$ associated with an eigenvalue $\lambda$, that is, $\mathbf{v} \neq \mathbf{0}$ and $A\mathbf{v} = \lambda\mathbf{v}$. Then

$$A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda(A\mathbf{v}) = \lambda(\lambda\mathbf{v}) = \lambda^2\mathbf{v}.$$

Therefore $\mathbf{v}$ is also an eigenvector of the matrix $A^2$ and the associated eigenvalue is $\lambda^2$. We already know that the matrix $A$ has eigenvalues $-1$, 1, and 3. It follows that $A^2$ has eigenvalues 1 and 9.

Since a $3\times3$ matrix can have up to 3 eigenvalues, we need an additional argument to show that 1 and 9 are the only eigenvalues of $A^2$. One reason is that the eigenvalue 1 has multiplicity 2.

**Problem 5.** Find a linear polynomial which is the best least squares fit to the following data:

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $f(x)$ | $-3$ | $-2$ | $1$ | $2$ | $5$ |

We are looking for a function $f(x) = c_1 + c_2 x$, where $c_1, c_2$ are unknown coefficients. The data of the problem give rise to an overdetermined system of linear equations in variables $c_1$ and $c_2$:

$$\begin{cases} c_1 - 2c_2 = -3, \\ c_1 - c_2 = -2, \\ c_1 = 1, \\ c_1 + c_2 = 2, \\ c_1 + 2c_2 = 5. \end{cases}$$

This system is inconsistent.

We can represent the system as a matrix equation $A\mathbf{c} = \mathbf{y}$, where

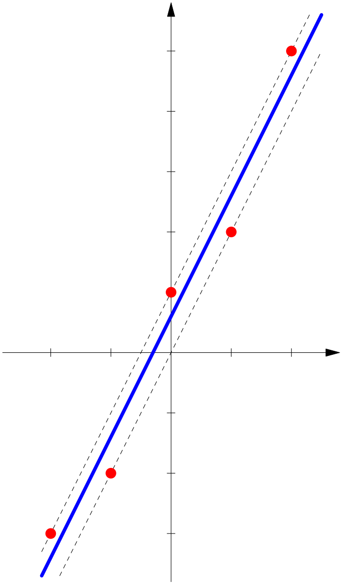
$$A = \begin{pmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \qquad \mathbf{y} = \begin{pmatrix} -3 \\ -2 \\ 1 \\ 2 \\ 5 \end{pmatrix}.$$

The least squares solution $\mathbf{c}$ of the above system is a solution of the normal system $A^T A\mathbf{c} = A^T\mathbf{y}$:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} -3 \\ -2 \\ 1 \\ 2 \\ 5 \end{pmatrix}$$

$$\iff \begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 20 \end{pmatrix} \iff \begin{cases} c_1 = 3/5 \\ c_2 = 2 \end{cases}$$

Thus the function $f(x) = \frac{3}{5} + 2x$ is the best least squares fit to the above data among linear polynomials.



**Problem 6.** Let $V$ be a subspace of $\mathbb{R}^4$ spanned by the vectors $\mathbf{x}_1 = (1,1,1,1)$ and $\mathbf{x}_2 = (1,0,3,0)$.
**(i)** Find an orthonormal basis for $V$.

First we apply the Gram-Schmidt orthogonalization process to vectors $\mathbf{x}_1, \mathbf{x}_2$ and obtain an orthogonal basis $\mathbf{v}_1, \mathbf{v}_2$ for the subspace $V$:

$$\mathbf{v}_1 = \mathbf{x}_1 = (1,1,1,1),$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = (1,0,3,0) - \frac{4}{4}(1,1,1,1) = (0,-1,2,-1).$$

Then we normalize vectors $\mathbf{v}_1, \mathbf{v}_2$ to obtain an orthonormal basis $\mathbf{w}_1, \mathbf{w}_2$ for $V$:

$$\|\mathbf{v}_1\| = 2 \implies \mathbf{w}_1 = \tfrac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \tfrac{1}{2}(1,1,1,1)$$

$$\|\mathbf{v}_2\| = \sqrt{6} \implies \mathbf{w}_2 = \tfrac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \tfrac{1}{\sqrt{6}}(0,-1,2,-1)$$

**Problem 6.** Let $V$ be a subspace of $\mathbb{R}^4$ spanned by the vectors $\mathbf{x}_1 = (1,1,1,1)$ and $\mathbf{x}_2 = (1,0,3,0)$.

**(ii)** Find an orthonormal basis for the orthogonal complement $V^\perp$.

Since the subspace $V$ is spanned by vectors $(1,1,1,1)$ and $(1,0,3,0)$, it is the row space of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 3 & 0 \end{pmatrix}.$$

Then the orthogonal complement $V^\perp$ is the nullspace of $A$. To find the nullspace, we convert the matrix $A$ to reduced row echelon form:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 3 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix}.$$

Hence a vector $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ belongs to $V^\perp$ if and only if

$$\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\iff \quad \begin{cases} x_1 + 3x_3 = 0 \\ x_2 - 2x_3 + x_4 = 0 \end{cases} \quad \iff \quad \begin{cases} x_1 = -3x_3 \\ x_2 = 2x_3 - x_4 \end{cases}$$

The general solution of the system is $(x_1, x_2, x_3, x_4) =$ $= (-3t, 2t - s, t, s) = t(-3, 2, 1, 0) + s(0, -1, 0, 1)$, where $t, s \in \mathbb{R}$.

It follows that $V^\perp$ is spanned by vectors $\mathbf{x}_3 = (0, -1, 0, 1)$ and $\mathbf{x}_4 = (-3, 2, 1, 0)$.

The vectors $\mathbf{x}_3 = (0, -1, 0, 1)$ and $\mathbf{x}_4 = (-3, 2, 1, 0)$ form a basis for the subspace $V^\perp$.

It remains to orthogonalize and normalize this basis:

$$\mathbf{v}_3 = \mathbf{x}_3 = (0, -1, 0, 1),$$

$$\mathbf{v}_4 = \mathbf{x}_4 - \frac{\mathbf{x}_4 \cdot \mathbf{v}_3}{\mathbf{v}_3 \cdot \mathbf{v}_3}\,\mathbf{v}_3 = (-3, 2, 1, 0) - \frac{-2}{2}(0, -1, 0, 1)$$
$$= (-3, 1, 1, 1),$$

$$\|\mathbf{v}_3\| = \sqrt{2} \implies \mathbf{w}_3 = \tfrac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \tfrac{1}{\sqrt{2}}(0, -1, 0, 1),$$

$$\|\mathbf{v}_4\| = \sqrt{12} = 2\sqrt{3} \implies \mathbf{w}_4 = \tfrac{\mathbf{v}_4}{\|\mathbf{v}_4\|} = \tfrac{1}{2\sqrt{3}}(-3, 1, 1, 1).$$

Thus the vectors $\mathbf{w}_3 = \frac{1}{\sqrt{2}}(0, -1, 0, 1)$ and $\mathbf{w}_4 = \frac{1}{2\sqrt{3}}(-3, 1, 1, 1)$ form an orthonormal basis for $V^\perp$.

**Problem 6.** Let $V$ be a subspace of $\mathbb{R}^4$ spanned by the vectors $\mathbf{x}_1 = (1, 1, 1, 1)$ and $\mathbf{x}_2 = (1, 0, 3, 0)$.

**(i)** Find an orthonormal basis for $V$.
**(ii)** Find an orthonormal basis for the orthogonal complement $V^\perp$.

*Alternative solution:* First we extend the set $\mathbf{x}_1, \mathbf{x}_2$ to a basis $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ for $\mathbb{R}^4$. Then we orthogonalize and normalize the latter. This yields an orthonormal basis $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4$ for $\mathbb{R}^4$.

By construction, $\mathbf{w}_1, \mathbf{w}_2$ is an orthonormal basis for $V$.
It follows that $\mathbf{w}_3, \mathbf{w}_4$ is an orthonormal basis for $V^\perp$.

The set $\mathbf{x}_1 = (1, 1, 1, 1)$, $\mathbf{x}_2 = (1, 0, 3, 0)$ can be extended to a basis for $\mathbb{R}^4$ by adding two vectors from the standard basis.

For example, we can add vectors $\mathbf{e}_3 = (0, 0, 1, 0)$ and $\mathbf{e}_4 = (0, 0, 0, 1)$. To show that $\mathbf{x}_1, \mathbf{x}_2, \mathbf{e}_3, \mathbf{e}_4$ is indeed a basis for $\mathbb{R}^4$, we check that the matrix whose rows are these vectors is nonsingular:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = - \begin{vmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1 \neq 0.$$

To orthogonalize the basis $\mathbf{x}_1, \mathbf{x}_2, \mathbf{e}_3, \mathbf{e}_4$, we apply the Gram-Schmidt process:

$$\mathbf{v}_1 = \mathbf{x}_1 = (1,1,1,1),$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = (1,0,3,0) - \tfrac{4}{4}(1,1,1,1) = (0,-1,2,-1),$$

$$\mathbf{v}_3 = \mathbf{e}_3 - \frac{\mathbf{e}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{e}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = (0,0,1,0) - \tfrac{1}{4}(1,1,1,1) -$$

$$-\tfrac{2}{6}(0,-1,2,-1) = \left(-\tfrac{1}{4}, \tfrac{1}{12}, \tfrac{1}{12}, \tfrac{1}{12}\right) = \tfrac{1}{12}(-3,1,1,1),$$

$$\mathbf{v}_4 = \mathbf{e}_4 - \frac{\mathbf{e}_4 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{e}_4 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 - \frac{\mathbf{e}_4 \cdot \mathbf{v}_3}{\mathbf{v}_3 \cdot \mathbf{v}_3}\mathbf{v}_3 = (0,0,0,1) -$$

$$-\tfrac{1}{4}(1,1,1,1) - \tfrac{-1}{6}(0,-1,2,-1) - \tfrac{1/12}{1/12} \cdot \tfrac{1}{12}(-3,1,1,1) =$$

$$= \left(0, -\tfrac{1}{2}, 0, \tfrac{1}{2}\right) = \tfrac{1}{2}(0,-1,0,1).$$

It remains to normalize vectors $\mathbf{v}_1 = (1, 1, 1, 1)$, $\mathbf{v}_2 = (0, -1, 2, -1)$, $\mathbf{v}_3 = \frac{1}{12}(-3, 1, 1, 1)$, $\mathbf{v}_4 = \frac{1}{2}(0, -1, 0, 1)$:

$$\|\mathbf{v}_1\| = 2 \implies \mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{2}(1, 1, 1, 1)$$

$$\|\mathbf{v}_2\| = \sqrt{6} \implies \mathbf{w}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{6}}(0, -1, 2, -1)$$

$$\|\mathbf{v}_3\| = \frac{1}{\sqrt{12}} = \frac{1}{2\sqrt{3}} \implies \mathbf{w}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{1}{2\sqrt{3}}(-3, 1, 1, 1)$$

$$\|\mathbf{v}_4\| = \frac{1}{\sqrt{2}} \implies \mathbf{w}_4 = \frac{\mathbf{v}_4}{\|\mathbf{v}_4\|} = \frac{1}{\sqrt{2}}(0, -1, 0, 1)$$

Thus $\mathbf{w}_1, \mathbf{w}_2$ is an orthonormal basis for $V$ while $\mathbf{w}_3, \mathbf{w}_4$ is an orthonormal basis for $V^\perp$.

**Problem 6.** Let $V$ be a subspace of $\mathbb{R}^4$ spanned by the vectors $\mathbf{x}_1 = (1,1,1,1)$ and $\mathbf{x}_2 = (1,0,3,0)$.

**(iii)** Find the distance from the vector $\mathbf{y} = (1,0,0,0)$ to the subspaces $V$ and $V^\perp$.

For any vector $\mathbf{y} \in \mathbb{R}^4$ the orthogonal projection of $\mathbf{y}$ onto the subspace $V$ is $\mathbf{p} = (\mathbf{y} \cdot \mathbf{w}_1)\mathbf{w}_1 + (\mathbf{y} \cdot \mathbf{w}_2)\mathbf{w}_2$ and the orthogonal projection of $\mathbf{y}$ onto $V^\perp$ is $\mathbf{o} = (\mathbf{y} \cdot \mathbf{w}_3)\mathbf{w}_3 + (\mathbf{y} \cdot \mathbf{w}_4)\mathbf{w}_4$.

Then the distance from $\mathbf{y}$ to $V$ is $\|\mathbf{y} - \mathbf{p}\| = \|\mathbf{o}\|$ and the distance from $\mathbf{y}$ to $V^\perp$ is $\|\mathbf{y} - \mathbf{o}\| = \|\mathbf{p}\|$.

In the case $\mathbf{y} = (1,0,0,0)$, we obtain

$$\mathbf{p} = \tfrac{1}{2} \cdot \tfrac{1}{2}(1,1,1,1) = \tfrac{1}{4}(1,1,1,1),$$

$$\mathbf{o} = \tfrac{-3}{2\sqrt{3}} \cdot \tfrac{1}{2\sqrt{3}}(-3,1,1,1) = \tfrac{1}{4}(3,-1,-1,-1).$$

Hence $\|\mathbf{o}\| = \frac{\sqrt{3}}{2}$ and $\|\mathbf{p}\| = \frac{1}{2}$.