MATH 311

Topics in Applied Mathematics I

Lecture 30:
Review of differential calculus.

Limit of a sequence

Definition. Sequence $x_1, x_2, x_3, \dots$ of real numbers is said to **converge** to a real number a if for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_n - a| < \varepsilon$ for all $n \geq N$. The number a is called the **limit** of $\{x_n\}$.

Notation: $\lim_{n\to\infty} x_n = a$, or $x_n \to a$ as $n \to \infty$.

Note that $d(x, y) = |x - y|$ is the distance between points x and y on the real line.

The condition $|x_n - a| < \varepsilon$ is equivalent to $x_n \in (a - \varepsilon, a + \varepsilon)$. The interval $(a - \varepsilon, a + \varepsilon)$ is called the ε-**neighborhood** of the point a. The convergence $x_n \to a$ means that any ε-neighborhood of a contains all but finitely many elements of the sequence $\{x_n\}$.

Limit of a function

Suppose $f : E \to \mathbb{R}$ is a function defined on a set $E \subset \mathbb{R}$.

Definition. We say that the function f **converges to a limit** $L \in \mathbb{R}$ at a point a if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$0 < |x - a| < \delta \text{ implies } |f(x) - L| < \varepsilon.$$

Notation: $L = \lim_{x \to a} f(x)$ or $f(x) \to L$ as $x \to a$.

Theorem Let I be an open interval containing a point $a \in \mathbb{R}$ and f be a function defined on $I \setminus \{a\}$. Then $f(x) \to L$ as $x \to a$ if and only if for any sequence $\{x_n\}$ of elements of $I \setminus \{a\}$,

$$\lim_{n \to \infty} x_n = a \quad \text{implies} \quad \lim_{n \to \infty} f(x_n) = L.$$

Continuity

Definition. Given a set $E \subset \mathbb{R}$, a function $f : E \to \mathbb{R}$, and a point $c \in E$, the function f is **continuous at** c if

$$f(c) = \lim_{x \to c} f(x).$$

That is, if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $|x - c| < \delta$ and $x \in E$ imply $|f(x) - f(c)| < \varepsilon$.

Theorem A function $f : E \to \mathbb{R}$ is continuous at a point $c \in E$ if and only if for any sequence $\{x_n\}$ of elements of E, $x_n \to c$ as $n \to \infty$ implies $f(x_n) \to f(c)$ as $n \to \infty$.

We say that the function f is **continuous on** a set $E_0 \subset E$ if f is continuous at every point $c \in E_0$. The function f is **continuous** if it is continuous on the entire domain E.

Topology of the real line

Definition. A sequence $\{x_n\}$ of real numbers is called a **Cauchy sequence** if for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_n - x_m| < \varepsilon$ whenever $n, m \geq N$.

Theorem (Cauchy) Any Cauchy sequence is convergent.

This property of $\mathbb{R}$ is called **completeness**.

Theorem (Bolzano-Weierstrass) Every bounded sequence of real numbers has a convergent subsequence.

This property of $\mathbb{R}$ is called **local compactness**.

A set $S \subset \mathbb{R}$ is called **compact** if any sequence of its elements has a subsequence converging to a limit in S. For example, any closed bounded interval $[a, b]$ is compact.

Extreme Value Theorem If $S \subset \mathbb{R}$ is compact, then any continuous function $f : S \to \mathbb{R}$ attains its extreme values on S.

The derivative

Definition. A real function f is said to be **differentiable** at a point $a \in \mathbb{R}$ if it is defined on an open interval containing a and the limit

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. The limit is denoted $f'(a)$ and called the **derivative** of f at a. An equivalent condition is

$$f(a+h) = f(a) + f'(a)h + r(h), \text{ where } \lim_{h \to 0} r(h)/h = 0.$$

If a function f is differentiable at a point a, then it is continuous at a.

Suppose that a function f is defined and differentiable on an interval I. Then the derivative of f can be regarded as a function on I. *Notation:* f', $\dot{f}$, $\dfrac{df}{dx}$, $D_x f$, $f^{(1)}$.

Differentiability theorems

Sum Rule If functions f and g are differentiable at a point $a \in \mathbb{R}$, then the sum $f + g$ is also differentiable at a. Moreover, $(f + g)'(a) = f'(a) + g'(a)$.

Homogeneous Rule If a function f is differentiable at a point $a \in \mathbb{R}$, then for any $r \in \mathbb{R}$ the scalar multiple rf is also differentiable at a. Moreover, $(rf)'(a) = rf'(a)$.

Difference Rule If functions f and g are differentiable at a point $a \in \mathbb{R}$, then the difference $f - g$ is also differentiable at a. Moreover, $(f - g)'(a) = f'(a) - g'(a)$.

Differentiability theorems

Product Rule If functions f and g are differentiable at a point $a \in \mathbb{R}$, then the product fg is also differentiable at a. Moreover, $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$.

Reciprocal Rule If a function f is differentiable at a point $a \in \mathbb{R}$ and $f(a) \neq 0$, then the function $1/f$ is also differentiable at a. Moreover, $(1/f)'(a) = -f'(a)/f^2(a)$.

Quotient Rule If functions f and g are differentiable at $a \in \mathbb{R}$ and $g(a) \neq 0$, then the quotient f/g is also differentiable at a. Moreover,

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}.$$

Differentiability theorems

Chain Rule If a function f is differentiable at a point $a \in \mathbb{R}$ and a function g is differentiable at $f(a)$, then the composition $g \circ f$ is differentiable at a. Moreover, $(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$.

Derivative of the inverse function Suppose f is an invertible continuous function. If f is differentiable at a point a and $f'(a) \neq 0$, then the inverse function is differentiable at the point $b = f(a)$ and, moreover,

$$(f^{-1})'(b) = \frac{1}{f'(a)}.$$

In the case $f'(a) = 0$, the inverse function f^{-1} is not differentiable at $f(a)$.

Properties of differentiable functions

Fermat's Theorem If a function f is differentiable at a point c of local extremum (maximum or minimum), then $f'(c) = 0$.

Rolle's Theorem If a function f is continuous on a closed interval $[a, b]$, differentiable on the open interval (a, b), and if $f(a) = f(b)$, then $f'(c) = 0$ for some $c \in (a, b)$.

Mean Value Theorem If a function f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that $f(b) - f(a) = f'(c)\,(b - a)$.

Problem. Find $\min_{x>0} x^x$.

The function $f(x) = x^x$ is well defined and positive on $(0, \infty)$. Hence

$$f(x) = e^{\log f(x)} = e^{\log x^x} = e^{x \log x}$$

for all $x > 0$. That is, $f(x) = g(h(x))$, where $h(x) = x \log x$ and $g(y) = e^y$. Using the Chain Rule and the Product Rule, we obtain

$$f'(x) = e^{x \log x}(x \log x)' = x^x \Big((x)' \log x + x(\log x)' \Big)$$
$$= x^x(\log x + 1).$$

It follows that $f'(x) < 0$ for $0 < x < 1/e$ and $f'(x) > 0$ for $x > 1/e$. Hence the function f is strictly decreasing on $(0, 1/e]$ and strictly increasing on $[1/e, \infty)$. Therefore

$$\min_{x>0} f(x) = f(1/e) = (1/e)^{1/e} = e^{-1/e}.$$

Vector-valued functions

Definition. Let $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \dots$ be a sequence of vectors in $\mathbb{R}^n$, $\mathbf{v}^{(k)} = (x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)})$. We say that the sequence converges to a vector $\mathbf{u} = (y_1, y_2, \dots, y_n)$ if $x_i^{(k)} \to y_i$ as $k \to \infty$, i.e., if each coordinate converges.

A vector-valued function $\mathbf{v} : X \to \mathbb{R}^n$ defined on a set $X \subset \mathbb{R}$ is essentially a collection of real-valued functions $f_i : X \to \mathbb{R}$, $1 \le i \le n$, such that $\mathbf{v}(t) = (f_1(t), f_2(t), \dots, f_n(t))$ for all $t \in X$.

We say that $\lim_{t\to a} \mathbf{v}(t) = \mathbf{u} = (y_1, y_2, \dots, y_n)$ if $\lim_{t\to a} f_i(t) = y_i$ for $1 \le i \le n$. Then the function $\mathbf{v}$ is continuous at a point $a \in X$ if each f_i is continuous at a.

Finally, we say that the function $\mathbf{v}$ is differentiable at a point a if each f_i is differentiable at a. The derivative is, by definition, $\mathbf{v}'(a) = (f_1'(a), f_2'(a), \dots, f_n'(a))$.

Differentiability theorems

Sum Rule If functions $\mathbf{v} : X \to \mathbb{R}^n$ and $\mathbf{w} : X \to \mathbb{R}^n$ are differentiable at a point $a \in \mathbb{R}$, then the sum $\mathbf{v} + \mathbf{w}$ is also differentiable at a. Moreover, $(\mathbf{v} + \mathbf{w})'(a) = \mathbf{v}'(a) + \mathbf{w}'(a)$.

Homogeneous Rule If a function $\mathbf{v} : X \to \mathbb{R}^n$ is differentiable at a point $a \in \mathbb{R}$, then for any $r \in \mathbb{R}$ the scalar multiple $r\mathbf{v}$ is also differentiable at a. Moreover, $(r\mathbf{v})'(a) = r\mathbf{v}'(a)$.

Difference Rule If functions $\mathbf{v} : X \to \mathbb{R}^n$ and $\mathbf{w} : X \to \mathbb{R}^n$ are differentiable at a point $a \in \mathbb{R}$, then the difference $\mathbf{v} - \mathbf{w}$ is also differentiable at a. Moreover, $(\mathbf{v} - \mathbf{w})'(a) = \mathbf{v}'(a) - \mathbf{w}'(a)$.

Differentiability theorems

Product Rule #1 If functions $f : X \to \mathbb{R}$ and $\mathbf{v} : X \to \mathbb{R}^n$ are differentiable at a point $a \in \mathbb{R}$, then the scalar multiple $f\mathbf{v}$ is also differentiable at a. Moreover, $(f\mathbf{v})'(a) = f'(a)\mathbf{v}(a) + f(a)\mathbf{v}'(a)$.

Product Rule #2 If functions $\mathbf{v} : X \to \mathbb{R}^n$ and $\mathbf{w} : X \to \mathbb{R}^n$ are differentiable at a point $a \in \mathbb{R}$, then the dot product $\mathbf{v} \cdot \mathbf{w}$ is also differentiable at a. Moreover, $(\mathbf{v} \cdot \mathbf{w})'(a) = \mathbf{v}'(a) \cdot \mathbf{w}(a) + \mathbf{v}(a) \cdot \mathbf{w}'(a)$.

Chain Rule If a function $f : X \to \mathbb{R}$ is differentiable at a point $a \in \mathbb{R}$ and a function $\mathbf{v} : Y \to \mathbb{R}^n$ is differentiable at $f(a)$, then the composition $\mathbf{v} \circ f$ is differentiable at a. Moreover, $(\mathbf{v} \circ f)'(a) = f'(a)\mathbf{v}'(f(a))$.

Matrix-valued functions

Definition. Let $A^{(1)}, A^{(2)}, \dots$ be a sequence of $m \times n$ matrices, $A^{(k)} = (a_{ij}^{(k)})$. We say that the sequence converges to an $m \times n$ matrix $B = (b_{ij})$ if $a_{ij}^{(k)} \to b_{ij}$ as $k \to \infty$, i.e., if each entry converges.

A matrix-valued function $A : X \to \mathcal{M}_{m,n}(\mathbb{R})$ defined on a set $X \subset \mathbb{R}$ is essentially a collection of mn real-valued functions $f_{ij} : X \to \mathbb{R}$ such that $A(t) = (f_{ij}(t))$ for all $t \in X$.

Limits, continuity, differentiability, and derivatives for such functions are defined in the same way as for vector-valued functions.

Some differentiability theorems

Sum Rule If functions $A : X \to \mathcal{M}_{m,n}(\mathbb{R})$ and $B : X \to \mathcal{M}_{m,n}(\mathbb{R})$ are differentiable at a point $a \in \mathbb{R}$, then the sum $A + B$ is also differentiable at a. Moreover, $(A + B)'(a) = A'(a) + B'(a)$.

Product Rule If functions $A : X \to \mathcal{M}_{m,n}(\mathbb{R})$ and $B : X \to \mathcal{M}_{n,k}(\mathbb{R})$ are differentiable at a point $a \in \mathbb{R}$, then the matrix product AB is also differentiable at a. Moreover, $(AB)'(a) = A'(a)B(a) + A(a)B'(a)$.

Chain Rule If a function $f : X \to \mathbb{R}$ is differentiable at a point $a \in \mathbb{R}$ and a function $A : X \to \mathcal{M}_{m,n}(\mathbb{R})$ is differentiable at $f(a)$, then the composition $A \circ f$ is differentiable at a. Moreover, $(A \circ f)'(a) = f'(a)A'(f(a))$.