MATH 311

Topics in Applied Mathematics I

Lecture 36:
Surface integrals (continued).
Gauss' theorem.
Stokes' theorem.

Surface integrals

Let $\mathbf{X} : D \to \mathbb{R}^3$ be a smooth parametrized surface, where $D \subset \mathbb{R}^2$ is a bounded region. Then for any continuous function $f : \mathbf{X}(D) \to \mathbb{R}$, the **scalar integral** of f over the surface $\mathbf{X}$ is

$$\iint_{\mathbf{X}} f\, dS = \iint_D f\left(\mathbf{X}(s,t)\right) \left\| \frac{\partial \mathbf{X}}{\partial s} \times \frac{\partial \mathbf{X}}{\partial t} \right\| ds\, dt.$$

For any continuous vector field $\mathbf{F} : \mathbf{X}(D) \to \mathbb{R}^3$, the **vector integral** of $\mathbf{F}$ along $\mathbf{X}$ is

$$\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}\left(\mathbf{X}(s,t)\right) \cdot \left(\frac{\partial \mathbf{X}}{\partial s} \times \frac{\partial \mathbf{X}}{\partial t} \right) ds\, dt.$$

Equivalently, $\displaystyle \iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} = \iint_D \begin{vmatrix} F_1 & F_2 & F_3 \\ \frac{\partial X_1}{\partial s} & \frac{\partial X_2}{\partial s} & \frac{\partial X_3}{\partial s} \\ \frac{\partial X_1}{\partial t} & \frac{\partial X_2}{\partial t} & \frac{\partial X_3}{\partial t} \end{vmatrix} ds\, dt.$

Surface integrals and reparametrization

Given two smooth parametrized surfaces $\mathbf{X} : D_1 \to \mathbb{R}^3$ and $\mathbf{Y} : D_2 \to \mathbb{R}^3$, we say that $\mathbf{Y}$ is a **smooth reparametrization** of $\mathbf{X}$ if there exists an invertible function $\mathbf{H} : D_2 \to D_1$ such that $\mathbf{Y} = \mathbf{X} \circ \mathbf{H}$ and both $\mathbf{H}$ and $\mathbf{H}^{-1}$ are smooth.

Theorem 1 Any scalar surface integral is invariant under smooth reparametrizations.

Any smooth parametrization of a surface defines an **orientation** on it (continuous, unit normal vector field $\mathbf{n}$).

Theorem 2 Any vector surface integral is invariant under smooth orientation-preserving reparametrizations and changes its sign under orientation-reversing reparametrizations.

As a consequence, we can define the integral of a function over a non-parametrized smooth surface and the integral of a vector field along a non-parametrized, oriented smooth surface.

Problem. Let C denote the closed cylinder with bottom given by $z = 0$, top given by $z = 4$, and lateral surface given by $x^2 + y^2 = 9$. We orient ∂C with outward normals. Find the integral of a vector field $\mathbf{F}(x, y, z) = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ along ∂C.

To evaluate the integral, we cut the boundary ∂C into three parts: the top, the bottom and the lateral surface.

The top of the cylinder is parametrized by $\mathbf{X}_{\mathrm{top}} : D \to \mathbb{R}^3$, $\mathbf{X}_{\mathrm{top}}(x, y) = (x, y, 4)$, where

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9\}.$$

The bottom is parametrized by $\mathbf{X}_{\mathrm{bot}} : D \to \mathbb{R}^3$, $\mathbf{X}_{\mathrm{bot}}(x, y) = (x, y, 0)$.

The lateral surface is parametrized by $\mathbf{X}_{\mathrm{lat}} : [0, 2\pi] \times [0, 4] \to \mathbb{R}^3$, $\mathbf{X}_{\mathrm{lat}}(\phi, z) = (3\cos\phi, 3\sin\phi, z)$.

We have $\frac{\partial \mathbf{X}_{\rm top}}{\partial x} = (1,0,0)$, $\frac{\partial \mathbf{X}_{\rm top}}{\partial y} = (0,1,0)$. Hence $\frac{\partial \mathbf{X}_{\rm top}}{\partial x} \times \frac{\partial \mathbf{X}_{\rm top}}{\partial y} = \mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$.

Since $\mathbf{X}_{\rm bot} = \mathbf{X}_{\rm top} - (0,0,4)$, we also have $\frac{\partial \mathbf{X}_{\rm bot}}{\partial x} = \mathbf{e}_1$, $\frac{\partial \mathbf{X}_{\rm bot}}{\partial y} = \mathbf{e}_2$, and $\frac{\partial \mathbf{X}_{\rm bot}}{\partial x} \times \frac{\partial \mathbf{X}_{\rm bot}}{\partial y} = \mathbf{e}_3$.

Further, $\frac{\partial \mathbf{X}_{\rm lat}}{\partial \phi} = (-3\sin\phi, 3\cos\phi, 0)$ and $\frac{\partial \mathbf{X}_{\rm lat}}{\partial z} = (0,0,1)$. Therefore

$$\frac{\partial \mathbf{X}_{\rm lat}}{\partial \phi} \times \frac{\partial \mathbf{X}_{\rm lat}}{\partial z} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ -3\sin\phi & 3\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = (3\cos\phi, 3\sin\phi, 0).$$

We observe that $\mathbf{X}_{\rm top}$ and $\mathbf{X}_{\rm lat}$ agree with the orientation of the surface C while $\mathbf{X}_{\rm bot}$ does not. It follows that

$$\oiint_{\partial C} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathbf{X}_{\rm top}} \mathbf{F} \cdot d\mathbf{S} - \iint_{\mathbf{X}_{\rm bot}} \mathbf{F} \cdot d\mathbf{S} + \iint_{\mathbf{X}_{\rm lat}} \mathbf{F} \cdot d\mathbf{S}.$$

Integrating the vector field $\mathbf{F} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ along each part of the boundary of C, we obtain:

$$\iint_{\mathbf{X}_{\mathrm{top}}} \mathbf{F}\cdot d\mathbf{S} = \iint_D (x, y, 4)\cdot(0, 0, 1)\, dx\, dy = \iint_D 4\, dx\, dy = 36\pi,$$

$$\iint_{\mathbf{X}_{\mathrm{bot}}} \mathbf{F}\cdot d\mathbf{S} = \iint_D (x, y, 0)\cdot(0, 0, 1)\, dx\, dy = \iint_D 0\, dx\, dy = 0,$$

$$\iint_{\mathbf{X}_{\mathrm{lat}}} \mathbf{F}\cdot d\mathbf{S} =$$

$$= \iint_{[0,2\pi]\times[0,4]} (3\cos\phi, 3\sin\phi, z)\cdot(3\cos\phi, 3\sin\phi, 0)\, d\phi\, dz$$

$$= \iint_{[0,2\pi]\times[0,4]} 9\, d\phi\, dz = 72\pi.$$

Thus $\displaystyle\oiint_{\partial C} \mathbf{F}\cdot d\mathbf{S} = 36\pi - 0 + 72\pi = 108\pi.$

Gauss's Theorem (a.k.a. Divergence Theorem in $\mathbb{R}^3$)

Theorem Let $D \subset \mathbb{R}^3$ be a closed, bounded region with piecewise smooth boundary ∂D (not necessarily connected) oriented by **outward** unit normals to D. Then for any smooth vector field $\mathbf{F}$ on D,

$$\oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{F} \, dV .$$

Corollary If a smooth vector field $\mathbf{F} : D \to \mathbb{R}^3$ has no divergence, $\nabla \cdot \mathbf{F} = 0$, then $\oiint_C \mathbf{F} \cdot d\mathbf{S} = 0$ for any closed, piecewise smooth surface C that bounds a subregion of D.

Gauss' Theorem

Proof in the case $D = [0,1]\times[0,1]\times[0,1]$ *and* $\mathbf{F} = (0,0,P)$*:*

$$\int_0^1 \frac{\partial P}{\partial z}(x,y,\zeta)\,d\zeta = P(x,y,1) - P(x,y,0)$$

for any $x, y \in [0,1]$ due to the Fundamental Theorem of Calculus. Integrating this equality over the unit square $Q = [0,1]\times[0,1]$, we obtain

$$\iiint_D \frac{\partial P}{\partial z}\,dV = \iint_Q P(x,y,1)\,dx\,dy - \iint_Q P(x,y,0)\,dx\,dy.$$

The first integral in the right-hand side equals the integral of the field $\mathbf{F}$ along the top face $Q\times\{1\}$ of the cube D (oriented by the upward unit normals). The second integral equals the integral of $\mathbf{F}$ along the bottom face $Q\times\{0\}$ (oriented likewise). Note that integrals of $\mathbf{F}$ along the other faces of D are 0 (since $\mathbf{F}$ is parallel to those faces). It follows that the entire right-hand side equals the integral of $\mathbf{F}$ along ∂D.

Problem. Let C denote the closed cylinder with bottom given by $z = 0$, top given by $z = 4$, and lateral surface given by $x^2 + y^2 = 9$. We orient ∂C with outward normals. Find the integral of a vector field $\mathbf{F}(x, y, z) = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ along ∂C.

This time, let us use Gauss' Theorem:

$$\oiint_{\partial C} \mathbf{F} \cdot d\mathbf{S} = \iiint_C \nabla \cdot \mathbf{F}\, dV$$

$$= \iiint_C \left(\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) \right) dx\, dy\, dz$$

$$= \iiint_C 3\, dx\, dy\, dz = 3\,\mathrm{volume}(C) = 108\pi.$$

Stokes's Theorem

Suppose S is an oriented surface in $\mathbb{R}^3$ bounded by an oriented curve ∂S. We say that ∂S is **oriented consistently with** S if, as one traverses ∂S, the surface S is on the left when looking down from the tip of $\mathbf{n}$, the unit normal vector indicating the orientation of S.

Theorem Let $S \subset \mathbb{R}^3$ be a bounded, piecewise smooth oriented surface with piecewise smooth boundary ∂S oriented consistently with S. Then for any smooth vector field $\mathbf{F}$ on S,

$$\iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s}.$$

Corollary If the surface S is closed (i.e., has no boundary), then for any smooth vector field $\mathbf{F}$ on S,

$$\iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = 0.$$

Example

Suppose that a bounded, piecewise smooth surface $S \subset \mathbb{R}^3$ is contained in the xy-coordinate plane, that is, $S = D \times \{0\}$ for a domain $D \subset \mathbb{R}^2$. We orient S by the upward unit normal vector $\mathbf{n} = (0, 0, 1)$ and orient the boundary $\partial S = \partial D \times \{0\}$ consistently with S. Further, suppose that $\mathbf{F}$ is a horizontal vector field, $\mathbf{F} = (M, N, 0)$. By Stokes' Theorem,

$$\iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s}.$$

Recall that $\iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = \iint_S \operatorname{curl}(\mathbf{F}) \cdot \mathbf{n}\, dS$. We obtain

$$\operatorname{curl}(\mathbf{F}) \cdot \mathbf{n} = \begin{vmatrix} 0 & 0 & 1 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.$$

It follows that this particular case of Stokes' Theorem is equivalent to Green's Theorem.