MATH 311

Topics in Applied Mathematics I

Lecture 8:
Elementary matrices.
Transpose of a matrix.
Determinants.

General results on inverse matrices

Theorem 1 Given an $n \times n$ matrix A, the following conditions are equivalent:

(i) A is invertible;
(ii) $\mathbf{x} = \mathbf{0}$ is the only solution of the matrix equation $A\mathbf{x} = \mathbf{0}$;
(iii) the matrix equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for any n-dimensional column vector $\mathbf{b}$;
(iv) the row echelon form of A has no zero rows;
(v) the reduced row echelon form of A is the identity matrix.

Theorem 2 Suppose that a sequence of elementary row operations converts a matrix A into the identity matrix.

Then the same sequence of operations converts the identity matrix into the inverse matrix A^{-1}.

Why does it work?

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ 2b_1 & 2b_2 & 2b_3 \\ c_1 & c_2 & c_3 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1+3a_1 & b_2+3a_2 & b_3+3a_3 \\ c_1 & c_2 & c_3 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{pmatrix}.$$

Proposition Any elementary row operation can be simulated as left multiplication by a certain matrix.

Elementary matrices

$$E = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & O & \\ & & 1 & & & & \\ & & & r & & & \\ & & & & 1 & & \\ & O & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \quad \text{row } \#i$$

To obtain the matrix EA from A, multiply the ith row by r. To obtain the matrix AE from A, multiply the ith column by r.

Elementary matrices

$$E = \begin{pmatrix} 1 & & & & \\ \vdots & \ddots & & O & \\ 0 & \cdots & 1 & & \\ \vdots & & \vdots & \ddots & \\ 0 & \cdots & r & \cdots & 1 & \\ \vdots & & \vdots & & \vdots & \ddots & \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{pmatrix} \quad \begin{matrix} \\ \\ \text{row } \#i \\ \\ \text{row } \#j \\ \\ \\ \end{matrix}$$

To obtain the matrix EA from A, add r times the ith row to the jth row. To obtain the matrix AE from A, add r times the jth column to the ith column.

Elementary matrices

$$E = \begin{pmatrix} 1 & & & & & O & \\ & \ddots & & & & & \\ & & 0 & \cdots & 1 & & \\ & & \vdots & \ddots & \vdots & & \\ & & 1 & \cdots & 0 & & \\ & & & & & \ddots & \\ & O & & & & & 1 \end{pmatrix} \begin{matrix} \\ \\ \text{row } \#i \\ \\ \text{row } \#j \\ \\ \\ \end{matrix}$$

To obtain the matrix EA from A, interchange the ith row with the jth row. To obtain AE from A, interchange the ith column with the jth column.

Why does it work? (continued)

Assume that a square matrix A can be converted to the identity matrix by a sequence of elementary row operations. Then $E_k E_{k-1} \dots E_2 E_1 A = I$, where $E_1, E_2, \dots, E_k$ are elementary matrices simulating those operations.

Applying the same sequence of operations to the identity matrix, we obtain the matrix

$$B = E_k E_{k-1} \dots E_2 E_1 I = E_k E_{k-1} \dots E_2 E_1.$$

Thus $BA = I$. Besides, B is invertible since elementary matrices are invertible (why?). It follows that $A = B^{-1}$, then $B = A^{-1}$.

Transpose of a matrix

Definition. Given a matrix A, the **transpose** of A, denoted A^T, is the matrix whose rows are columns of A (and whose columns are rows of A). That is, if $A = (a_{ij})$ then $A^T = (b_{ij})$, where $b_{ij} = a_{ji}$.

Examples. $$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix},$$

$$\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}^T = (7, 8, 9), \qquad \begin{pmatrix} 4 & 7 \\ 7 & 0 \end{pmatrix}^T = \begin{pmatrix} 4 & 7 \\ 7 & 0 \end{pmatrix}.$$

Properties of transposes:

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(rA)^T = rA^T$
- $(AB)^T = B^T A^T$
- $(A_1 A_2 \ldots A_k)^T = A_k^T \ldots A_2^T A_1^T$
- $(A^{-1})^T = (A^T)^{-1}$

Definition. A square matrix A is said to be **symmetric** if $A^T = A$.

For example, any diagonal matrix is symmetric.

Proposition For any square matrix A the matrices $B = AA^T$ and $C = A + A^T$ are symmetric.

Proof:

$$B^T = (AA^T)^T = (A^T)^T A^T = AA^T = B,$$

$$C^T = (A + A^T)^T = A^T + (A^T)^T = A^T + A = C.$$

Determinants

Determinant is a scalar assigned to each square matrix.

Notation. The determinant of a matrix $A = (a_{ij})_{1 \le i,j \le n}$ is denoted $\det A$ or

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

Principal property: $\det A \neq 0$ if and only if a system of linear equations with the coefficient matrix A has a unique solution. Equivalently, $\det A \neq 0$ if and only if the matrix A is invertible.

Definition in low dimensions

Definition. $\det(a) = a$, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

$$+: \begin{pmatrix} \boxed{*} & * & * \\ * & \boxed{*} & * \\ * & * & \boxed{*} \end{pmatrix}, \begin{pmatrix} * & \boxed{*} & * \\ * & * & \boxed{*} \\ \boxed{*} & * & * \end{pmatrix}, \begin{pmatrix} * & * & \boxed{*} \\ \boxed{*} & * & * \\ * & \boxed{*} & * \end{pmatrix}.$$

$$-: \begin{pmatrix} * & * & \boxed{*} \\ * & \boxed{*} & * \\ \boxed{*} & * & * \end{pmatrix}, \begin{pmatrix} * & \boxed{*} & * \\ \boxed{*} & * & * \\ * & * & \boxed{*} \end{pmatrix}, \begin{pmatrix} \boxed{*} & * & * \\ * & * & \boxed{*} \\ * & \boxed{*} & * \end{pmatrix}.$$

Examples: 2×2 matrices

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, \qquad \begin{vmatrix} 3 & 0 \\ 0 & -4 \end{vmatrix} = -12,$$

$$\begin{vmatrix} -2 & 5 \\ 0 & 3 \end{vmatrix} = -6, \qquad \begin{vmatrix} 7 & 0 \\ 5 & 2 \end{vmatrix} = 14,$$

$$\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = 1, \qquad \begin{vmatrix} 0 & 0 \\ 4 & 1 \end{vmatrix} = 0,$$

$$\begin{vmatrix} -1 & 3 \\ -1 & 3 \end{vmatrix} = 0, \qquad \begin{vmatrix} 2 & 1 \\ 8 & 4 \end{vmatrix} = 0.$$

Examples: 3×3 matrices

$$\begin{vmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{vmatrix} = 3\cdot 0\cdot 0 + (-2)\cdot 1\cdot(-2) + 0\cdot 1\cdot 3 -$$

$$- 0\cdot 0\cdot(-2) - (-2)\cdot 1\cdot 0 - 3\cdot 1\cdot 3 = 4 - 9 = -5,$$

$$\begin{vmatrix} 1 & 4 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{vmatrix} = 1\cdot 2\cdot 3 + 4\cdot 5\cdot 0 + 6\cdot 0\cdot 0 -$$

$$- 6\cdot 2\cdot 0 - 4\cdot 0\cdot 3 - 1\cdot 5\cdot 0 = 1\cdot 2\cdot 3 = 6.$$

General definition

The general definition of the determinant is quite complicated as there is no simple explicit formula.

There are several approaches to defining determinants.

Approach 1 (original): an explicit (but very complicated) formula.

Approach 2 (axiomatic): we formulate properties that the determinant should have.

Approach 3 (inductive): the determinant of an $n \times n$ matrix is defined in terms of determinants of certain $(n-1)\times(n-1)$ matrices.

Axiomatic definition

$\mathcal{M}_{n,n}(\mathbb{R})$: the set of $n \times n$ matrices with real entries.

Theorem There exists a unique function $\det : \mathcal{M}_{n,n}(\mathbb{R}) \to \mathbb{R}$ (called the determinant) with the following properties:

(D1) if a row of a matrix is multiplied by a scalar r, the determinant is also multiplied by r;

(D2) if we add a row of a matrix multiplied by a scalar to another row, the determinant remains the same;

(D3) if we interchange two rows of a matrix, the determinant changes its sign;

(D4) $\det I = 1$.