MATH 311 Topics in Applied Mathematics I Lecture 8: Elementary matrices. Transpose of a matrix. Determinants.

#### General results on inverse matrices

**Theorem 1** Given an  $n \times n$  matrix A, the following conditions are equivalent:

(i) A is invertible;

(ii)  $\mathbf{x} = \mathbf{0}$  is the only solution of the matrix equation  $A\mathbf{x} = \mathbf{0}$ ; (iii) the matrix equation  $A\mathbf{x} = \mathbf{b}$  has a unique solution for any *n*-dimensional column vector **b**;

(iv) the row echelon form of A has no zero rows;

(v) the reduced row echelon form of A is the identity matrix.

**Theorem 2** Suppose that a sequence of elementary row operations converts a matrix *A* into the identity matrix.

Then the same sequence of operations converts the identity matrix into the inverse matrix  $A^{-1}$ .

Why does it work?

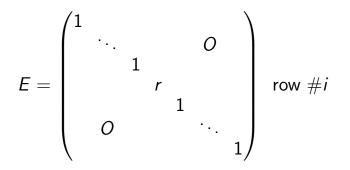
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ 2b_1 & 2b_2 & 2b_3 \\ c_1 & c_2 & c_3 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ c_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 + 3a_1 & b_2 + 3a_2 & b_3 + 3a_3 \\ c_1 & c_2 & c_3 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 + 3a_1 & b_2 + 3a_2 & b_3 + 3a_3 \\ c_1 & c_2 & c_3 \end{pmatrix},$$

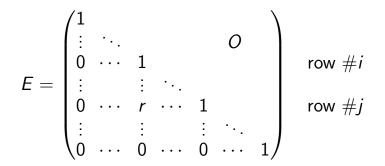
**Proposition** Any elementary row operation can be simulated as left multiplication by a certain matrix.

#### **Elementary matrices**



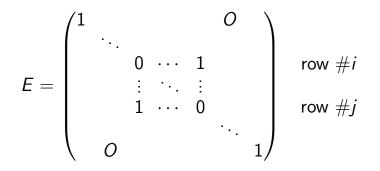
To obtain the matrix EA from A, multiply the *i*th row by r. To obtain the matrix AE from A, multiply the *i*th column by r.

#### **Elementary matrices**



To obtain the matrix EA from A, add r times the *i*th row to the *j*th row. To obtain the matrix AE from A, add r times the *j*th column to the *i*th column.

#### **Elementary matrices**



To obtain the matrix EA from A, interchange the *i*th row with the *j*th row. To obtain AE from A, interchange the *i*th column with the *j*th column.

# Why does it work? (continued)

Assume that a square matrix A can be converted to the identity matrix by a sequence of elementary row operations. Then  $E_k E_{k-1} \dots E_2 E_1 A = I$ , where  $E_1, E_2, \dots, E_k$  are elementary matrices simulating those operations.

Applying the same sequence of operations to the identity matrix, we obtain the matrix

$$B = E_k E_{k-1} \dots E_2 E_1 I = E_k E_{k-1} \dots E_2 E_1.$$

Thus BA = I. Besides, *B* is invertible since elementary matrices are invertible (why?). It follows that  $A = B^{-1}$ , then  $B = A^{-1}$ .

#### Transpose of a matrix

Definition. Given a matrix A, the **transpose** of A, denoted  $A^{T}$ , is the matrix whose rows are columns of A (and whose columns are rows of A). That is, if  $A = (a_{ij})$  then  $A^{T} = (b_{ij})$ , where  $b_{ij} = a_{ji}$ .

Examples. 
$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^{T} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$
,  
 $\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}^{T} = (7, 8, 9), \qquad \begin{pmatrix} 4 & 7 \\ 7 & 0 \end{pmatrix}^{T} = \begin{pmatrix} 4 & 7 \\ 7 & 0 \end{pmatrix}$ 

# Properties of transposes:

• 
$$(A^T)^T = A$$

•  $(A+B)^T = A^T + B^T$ 

• 
$$(rA)^T = rA^T$$

- $(AB)^T = B^T A^T$
- $(A_1A_2\ldots A_k)^T = A_k^T\ldots A_2^TA_1^T$

• 
$$(A^{-1})^T = (A^T)^{-1}$$

Definition. A square matrix A is said to be symmetric if  $A^T = A$ .

For example, any diagonal matrix is symmetric.

**Proposition** For any square matrix A the matrices  $B = AA^T$  and  $C = A + A^T$  are symmetric.

Proof:

$$B^{T} = (AA^{T})^{T} = (A^{T})^{T}A^{T} = AA^{T} = B,$$
  
 $C^{T} = (A + A^{T})^{T} = A^{T} + (A^{T})^{T} = A^{T} + A = C.$ 

## Determinants

Determinant is a scalar assigned to each square matrix.

*Notation.* The determinant of a matrix  $A = (a_{ii})_{1 \le i, i \le n}$  is denoted det A or

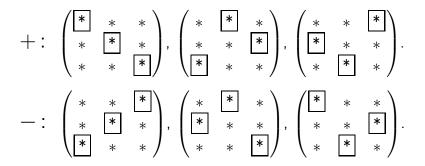
I					
	$a_{11}$	$a_{12}$	• • •	a <sub>1n</sub>	
	<b>a</b> <sub>21</sub>	<i>a</i> <sub>22</sub>	• • •	a <sub>2n</sub>	
	÷	÷	•••	÷	•
	a <sub>n1</sub>	a <sub>n2</sub>	•••	a <sub>nn</sub>	

**Principal property:** det  $A \neq 0$  if and only if a system of linear equations with the coefficient matrix A has a unique solution. Equivalently, det  $A \neq 0$  if and only if the matrix A is invertible.

## Definition in low dimensions

Definition. det (a) = a, 
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$
 = ad - bc,  $\begin{vmatrix} a_{11} & a_{12} & a_{13} \end{vmatrix}$ 

 $\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ -a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$ 



## **Examples: 2**×2 matrices

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, \qquad \begin{vmatrix} 3 & 0 \\ 0 & -4 \end{vmatrix} = -12,$$
$$\begin{vmatrix} -2 & 5 \\ 0 & 3 \end{vmatrix} = -6, \qquad \begin{vmatrix} 7 & 0 \\ 5 & 2 \end{vmatrix} = 14,$$
$$\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = 1, \qquad \begin{vmatrix} 0 & 0 \\ 4 & 1 \end{vmatrix} = 0,$$
$$\begin{vmatrix} -1 & 3 \\ -1 & 3 \end{vmatrix} = 0, \qquad \begin{vmatrix} 2 & 1 \\ 8 & 4 \end{vmatrix} = 0.$$

#### **Examples:** 3×3 matrices

$$\begin{vmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{vmatrix} = 3 \cdot 0 \cdot 0 + (-2) \cdot 1 \cdot (-2) + 0 \cdot 1 \cdot 3 - \\ -0 \cdot 0 \cdot (-2) - (-2) \cdot 1 \cdot 0 - 3 \cdot 1 \cdot 3 = 4 - 9 = -5, \\\begin{vmatrix} 1 & 4 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{vmatrix} = 1 \cdot 2 \cdot 3 + 4 \cdot 5 \cdot 0 + 6 \cdot 0 \cdot 0 - \end{vmatrix}$$

 $-6 \cdot 2 \cdot 0 - 4 \cdot 0 \cdot 3 - 1 \cdot 5 \cdot 0 = 1 \cdot 2 \cdot 3 = 6.$ 

# **General definition**

The general definition of the determinant is quite complicated as there is no simple explicit formula.

There are several approaches to defining determinants.

**Approach 1 (original):** an explicit (but very complicated) formula.

**Approach 2 (axiomatic):** we formulate properties that the determinant should have.

**Approach 3 (inductive):** the determinant of an  $n \times n$  matrix is defined in terms of determinants of certain  $(n-1) \times (n-1)$  matrices.

# **Axiomatic definition**

 $\mathcal{M}_{n,n}(\mathbb{R})$ : the set of  $n \times n$  matrices with real entries.

**Theorem** There exists a unique function det :  $\mathcal{M}_{n,n}(\mathbb{R}) \to \mathbb{R}$  (called the determinant) with the following properties:

**(D1)** if a row of a matrix is multiplied by a scalar *r*, the determinant is also multiplied by *r*;

**(D2)** if we add a row of a matrix multiplied by a scalar to another row, the determinant remains the same;

**(D3)** if we interchange two rows of a matrix, the determinant changes its sign;

**(D4)** det I = 1.