

MATH 311

Topics in Applied Mathematics I

**Lecture 22:**  
**Eigenvalues and eigenvectors**  
**of a linear operator.**

## Eigenvalues and eigenvectors of a matrix

*Definition.* Let  $A$  be an  $n \times n$  matrix. A number  $\lambda \in \mathbb{R}$  is called an **eigenvalue** of the matrix  $A$  if  $A\mathbf{v} = \lambda\mathbf{v}$  for a nonzero column vector  $\mathbf{v} \in \mathbb{R}^n$ .

The vector  $\mathbf{v}$  is called an **eigenvector** of  $A$  belonging to (or associated with) the eigenvalue  $\lambda$ .

If  $\lambda$  is an eigenvalue of  $A$  then the nullspace  $N(A - \lambda I)$ , which is nontrivial, is called the **eigenspace** of  $A$  corresponding to  $\lambda$ . The eigenspace consists of all eigenvectors belonging to the eigenvalue  $\lambda$  plus the zero vector.

## Characteristic equation

*Definition.* Given a square matrix  $A$ , the equation  $\det(A - \lambda I) = 0$  is called the **characteristic equation** of  $A$ .

Eigenvalues  $\lambda$  of  $A$  are roots of the characteristic equation.

If  $A$  is an  $n \times n$  matrix then  $p(\lambda) = \det(A - \lambda I)$  is a polynomial of degree  $n$ . It is called the **characteristic polynomial** of  $A$ .

**Theorem** Any  $n \times n$  matrix has at most  $n$  eigenvalues.

*Example.*  $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ .

Characteristic equation:  $\begin{vmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{vmatrix} = 0$ .

$$(2 - \lambda)^2 - 1 = 0 \implies \lambda_1 = 1, \lambda_2 = 3.$$

$$(A - I)\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff x + y = 0.$$

The general solution is  $(-t, t) = t(-1, 1)$ ,  $t \in \mathbb{R}$ .

Thus  $\mathbf{v}_1 = (-1, 1)$  is an eigenvector associated with the eigenvalue 1. The corresponding eigenspace is the line spanned by  $\mathbf{v}_1$ .

$$(A - 3I)\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\iff \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff x - y = 0.$$

The general solution is  $(t, t) = t(1, 1)$ ,  $t \in \mathbb{R}$ .

Thus  $\mathbf{v}_2 = (1, 1)$  is an eigenvector associated with the eigenvalue 3. The corresponding eigenspace is the line spanned by  $\mathbf{v}_2$ .

*Summary.*  $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ .

- The matrix  $A$  has two eigenvalues: 1 and 3.
- The eigenspace of  $A$  associated with the eigenvalue 1 is the line  $t(-1, 1)$ .
- The eigenspace of  $A$  associated with the eigenvalue 3 is the line  $t(1, 1)$ .
- Eigenvectors  $\mathbf{v}_1 = (-1, 1)$  and  $\mathbf{v}_2 = (1, 1)$  of the matrix  $A$  form a basis for  $\mathbb{R}^2$ .
- Geometrically, the mapping  $\mathbf{x} \mapsto A\mathbf{x}$  is a stretch by a factor of 3 away from the line  $x + y = 0$  in the orthogonal direction.

*Example.*  $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$

Characteristic equation:

$$\begin{vmatrix} 1 - \lambda & 1 & -1 \\ 1 & 1 - \lambda & 1 \\ 0 & 0 & 2 - \lambda \end{vmatrix} = 0.$$

Expand the determinant by the 3rd row:

$$(2 - \lambda) \begin{vmatrix} 1 - \lambda & 1 \\ 1 & 1 - \lambda \end{vmatrix} = 0.$$

$$\begin{aligned} ((1 - \lambda)^2 - 1)(2 - \lambda) &= 0 \iff -\lambda(2 - \lambda)^2 = 0 \\ \implies \lambda_1 &= 0, \lambda_2 = 2. \end{aligned}$$

$$A\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Convert the matrix to reduced row echelon form:

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A\mathbf{x} = \mathbf{0} \iff \begin{cases} x + y = 0, \\ z = 0. \end{cases}$$

The general solution is  $(-t, t, 0) = t(-1, 1, 0)$ ,  $t \in \mathbb{R}$ . Thus  $\mathbf{v}_1 = (-1, 1, 0)$  is an eigenvector associated with the eigenvalue 0. The corresponding eigenspace is the line spanned by  $\mathbf{v}_1$ .



$$(A - 2I)\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff x - y + z = 0.$$

The general solution is  $x = t - s$ ,  $y = t$ ,  $z = s$ , where  $t, s \in \mathbb{R}$ . Equivalently,

$$\mathbf{x} = (t - s, t, s) = t(1, 1, 0) + s(-1, 0, 1).$$

Thus  $\mathbf{v}_2 = (1, 1, 0)$  and  $\mathbf{v}_3 = (-1, 0, 1)$  are eigenvectors associated with the eigenvalue 2.

The corresponding eigenspace is the plane spanned by  $\mathbf{v}_2$  and  $\mathbf{v}_3$ .

Summary.  $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ .

- The matrix  $A$  has two eigenvalues: 0 and 2.
- The eigenvalue 0 is *simple*: the corresponding eigenspace is a line.
- The eigenvalue 2 is of *multiplicity* 2: the corresponding eigenspace is a plane.
- Eigenvectors  $\mathbf{v}_1 = (-1, 1, 0)$ ,  $\mathbf{v}_2 = (1, 1, 0)$ , and  $\mathbf{v}_3 = (-1, 0, 1)$  of the matrix  $A$  form a basis for  $\mathbb{R}^3$ .
- Geometrically, the map  $\mathbf{x} \mapsto A\mathbf{x}$  is the projection on the plane  $\text{Span}(\mathbf{v}_2, \mathbf{v}_3)$  along the lines parallel to  $\mathbf{v}_1$  with the subsequent scaling by a factor of 2.

## Eigenvalues and eigenvectors of an operator

*Definition.* Let  $V$  be a vector space and  $L : V \rightarrow V$  be a linear operator. A number  $\lambda$  is called an **eigenvalue** of the operator  $L$  if  $L(\mathbf{v}) = \lambda\mathbf{v}$  for a nonzero vector  $\mathbf{v} \in V$ . The vector  $\mathbf{v}$  is called an **eigenvector** of  $L$  associated with the eigenvalue  $\lambda$ . (If  $V$  is a functional space then eigenvectors are also called **eigenfunctions**.)

If  $V = \mathbb{R}^n$  then the linear operator  $L$  is given by  $L(\mathbf{x}) = A\mathbf{x}$ , where  $A$  is an  $n \times n$  matrix.

In this case, eigenvalues and eigenvectors of the operator  $L$  are precisely eigenvalues and eigenvectors of the matrix  $A$ .

Suppose  $L : V \rightarrow V$  is a linear operator on a **finite-dimensional** vector space  $V$ .

Let  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$  be a basis for  $V$  and  $g : V \rightarrow \mathbb{R}^n$  be the corresponding coordinate mapping. Let  $A$  be the matrix of  $L$  with respect to this basis. Then

$$L(\mathbf{v}) = \lambda \mathbf{v} \iff A g(\mathbf{v}) = \lambda g(\mathbf{v}).$$

Hence the eigenvalues of  $L$  coincide with those of the matrix  $A$ . Moreover, the associated eigenvectors of  $A$  are coordinates of the eigenvectors of  $L$ .

*Definition.* The characteristic polynomial  $p(\lambda) = \det(A - \lambda I)$  of the matrix  $A$  is called the **characteristic polynomial** of the operator  $L$ .

Then eigenvalues of  $L$  are roots of its characteristic polynomial.

**Theorem.** The characteristic polynomial of the operator  $L$  is well defined. That is, it does not depend on the choice of a basis.

*Proof:* Let  $B$  be the matrix of  $L$  with respect to a different basis  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ . Then  $A = UBU^{-1}$ , where  $U$  is the transition matrix from the basis  $\mathbf{v}_1, \dots, \mathbf{v}_n$  to  $\mathbf{u}_1, \dots, \mathbf{u}_n$ . We have to show that  $\det(A - \lambda I) = \det(B - \lambda I)$  for all  $\lambda \in \mathbb{R}$ . We obtain

$$\begin{aligned}\det(A - \lambda I) &= \det(UBU^{-1} - \lambda I) \\ &= \det(UBU^{-1} - U(\lambda I)U^{-1}) = \det(U(B - \lambda I)U^{-1}) \\ &= \det(U) \det(B - \lambda I) \det(U^{-1}) = \det(B - \lambda I).\end{aligned}$$

## Eigenspaces

Let  $L : V \rightarrow V$  be a linear operator.

For any  $\lambda \in \mathbb{R}$ , let  $V_\lambda$  denotes the set of all solutions of the equation  $L(\mathbf{x}) = \lambda\mathbf{x}$ .

Then  $V_\lambda$  is a *subspace* of  $V$  since  $V_\lambda$  is the *kernel* of a linear operator given by  $\mathbf{x} \mapsto L(\mathbf{x}) - \lambda\mathbf{x}$ .

$V_\lambda$  minus the zero vector is the set of all eigenvectors of  $L$  associated with the eigenvalue  $\lambda$ . In particular,  $\lambda \in \mathbb{R}$  is an eigenvalue of  $L$  if and only if  $V_\lambda \neq \{\mathbf{0}\}$ .

If  $V_\lambda \neq \{\mathbf{0}\}$  then it is called the **eigenspace** of  $L$  corresponding to the eigenvalue  $\lambda$ .

*Example.*  $V = C^\infty(\mathbb{R})$ ,  $D : V \rightarrow V$ ,  $Df = f'$ .

A function  $f \in C^\infty(\mathbb{R})$  is an eigenfunction of the operator  $D$  belonging to an eigenvalue  $\lambda$  if  $f'(x) = \lambda f(x)$  for all  $x \in \mathbb{R}$ .

It follows that  $f(x) = ce^{\lambda x}$ , where  $c$  is a nonzero constant.

Thus each  $\lambda \in \mathbb{R}$  is an eigenvalue of  $D$ .

The corresponding eigenspace is spanned by  $e^{\lambda x}$ .

*Example.*  $V = C^\infty(\mathbb{R})$ ,  $L : V \rightarrow V$ ,  $Lf = f''$ .

$$Lf = \lambda f \iff f''(x) - \lambda f(x) = 0 \text{ for all } x \in \mathbb{R}.$$

It follows that each  $\lambda \in \mathbb{R}$  is an eigenvalue of  $L$  and the corresponding eigenspace  $V_\lambda$  is two-dimensional. Note that  $L = D^2$ , hence  $Df = \mu f \implies Lf = \mu^2 f$ .

If  $\lambda > 0$  then  $V_\lambda = \text{Span}(e^{\mu x}, e^{-\mu x})$ , where  $\mu = \sqrt{\lambda}$ .

If  $\lambda < 0$  then  $V_\lambda = \text{Span}(\sin(\mu x), \cos(\mu x))$ , where  $\mu = \sqrt{-\lambda}$ .

If  $\lambda = 0$  then  $V_\lambda = \text{Span}(1, x)$ .



**Theorem** If  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  are eigenvectors of a linear operator  $L$  associated with distinct eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_k$ , then  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  are linearly independent.

**Corollary 1** If  $\lambda_1, \lambda_2, \dots, \lambda_k$  are distinct real numbers, then the functions  $e^{\lambda_1 x}, e^{\lambda_2 x}, \dots, e^{\lambda_k x}$  are linearly independent.

*Proof:* Consider a linear operator  $D : C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R})$  given by  $Df = f'$ . Then  $e^{\lambda_1 x}, \dots, e^{\lambda_k x}$  are eigenfunctions of  $D$  associated with distinct eigenvalues  $\lambda_1, \dots, \lambda_k$ . By the theorem, the eigenfunctions are linearly independent.

**Corollary 2** If  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  are eigenvectors of a matrix  $A$  associated with distinct eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_k$ , then  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  are linearly independent.

**Corollary 3** Let  $A$  be an  $n \times n$  matrix such that the characteristic equation  $\det(A - \lambda I) = 0$  has  $n$  distinct real roots. Then  $\mathbb{R}^n$  has a basis consisting of eigenvectors of  $A$ .

*Proof:* Let  $\lambda_1, \lambda_2, \dots, \lambda_n$  be distinct real roots of the characteristic equation. Any  $\lambda_i$  is an eigenvalue of  $A$ , hence there is an associated eigenvector  $\mathbf{v}_i$ . By Corollary 2, vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  are linearly independent. Therefore they form a basis for  $\mathbb{R}^n$ .