MATH 311

Topics in Applied Mathematics I

Lecture 35:
Conservative vector fields.
Area of a surface.
Surface integrals.

Conservative vector fields

Let R be an open region in $\mathbb{R}^n$ such that any two points in R can be connected by a continuous path. Such regions are called **(arcwise) connected**.

Definition. A continuous vector field $\mathbf{F} : R \to \mathbb{R}^n$ is called **conservative** if $\displaystyle\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_{C_2} \mathbf{F} \cdot d\mathbf{s}$ for any two simple, piecewise smooth, oriented curves $C_1, C_2 \subset R$ with the same initial and terminal points.

An equivalent condition is that $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{s} = 0$ for any piecewise smooth closed curve $C \subset R$.

Conservative vector fields

Theorem The vector field $\mathbf{F}$ is conservative if and only if it is a gradient field, that is, $\mathbf{F} = \nabla f$ for some function $f : R \to \mathbb{R}$. If this is the case, then

$$\int_C \mathbf{F} \cdot d\mathbf{s} = f(B) - f(A)$$

for any piecewise smooth, oriented curve $C \subset R$ that connects the point A to the point B.

Remark. In the case $\mathbf{F}$ is a force field, conservativity means that energy is conserved. Moreover, in this case the function f is the potential energy.

Test of conservativity

Theorem If a smooth field $\mathbf{F} = (F_1, F_2, \ldots, F_n)$ is conservative in a region $R \subset \mathbb{R}^n$, then the Jacobian matrix $\dfrac{\partial(F_1, F_2, \ldots, F_n)}{\partial(x_1, x_2, \ldots, x_n)}$ is symmetric everywhere in R, that is,

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i} \text{ for } i \neq j.$$

Indeed, if the field $\mathbf{F}$ is conservative, then $\mathbf{F} = \nabla f$ for some smooth function $f : R \to \mathbb{R}$. It follows that the Jacobian matrix of $\mathbf{F}$ is the **Hessian matrix** of f, that is, the matrix of second-order partial derivatives: $\dfrac{\partial F_i}{\partial x_j} = \dfrac{\partial^2 f}{\partial x_j\, \partial x_i}$.

Remark. The converse of the theorem holds provided that the region R is **simply-connected**, which means that any closed path in R can be continuously shrunk within R to a point.

Finding scalar potential

Example. $\mathbf{F}(x, y) = (2xy^3 + 3y\cos 3x,\ 3x^2y^2 + \sin 3x)$.

The vector field $\mathbf{F}$ is conservative if $\partial F_1/\partial y = \partial F_2/\partial x$.

$\dfrac{\partial F_1}{\partial y} = 6xy^2 + 3\cos 3x,\quad \dfrac{\partial F_2}{\partial x} = 6xy^2 + 3\cos 3x.$

Thus $\mathbf{F} = \nabla f$ for some function f (**scalar potential** of $\mathbf{F}$),

that is, $\dfrac{\partial f}{\partial x} = 2xy^3 + 3y\cos 3x,\quad \dfrac{\partial f}{\partial y} = 3x^2y^2 + \sin 3x.$

Integrating the second equality by y, we get

$$f(x, y) = \int (3x^2y^2 + \sin 3x)\, dy = x^2y^3 + y\sin 3x + g(x).$$

Substituting this into the first equality, we obtain that $2xy^3 + 3y\cos 3x + g'(x) = 2xy^3 + 3y\cos 3x$. Hence $g'(x) = 0$ so that $g(x) = c$, a constant. Then $f(x, y) = x^2y^3 + y\sin 3x + c$.

Surface

Suppose D_1 and D_2 are domains in $\mathbb{R}^3$ and $\mathbf{T} : D_1 \to D_2$ is an invertible map such that both $\mathbf{T}$ and $\mathbf{T}^{-1}$ are smooth. Then we say that $\mathbf{T}$ defines **curvilinear coordinates** in D_1.

Definition. A nonempty set $S \subset \mathbb{R}^3$ is called a **smooth surface** if for every point $\mathbf{p} \in S$ there exist curvilinear coordinates $\mathbf{T} : D_1 \to D_2$ in a neighborhood of $\mathbf{p}$ such that $\mathbf{T}(\mathbf{p}) = \mathbf{0}$ and either $\mathbf{T}(S \cap D_1) = \{(x, y, z) \in D_2 \mid z = 0\}$ or $\mathbf{T}(S \cap D_1) = \{(x, y, z) \in D_2 \mid z = 0,\ y \geq 0\}$. In the first case, $\mathbf{p}$ is called an **interior point** of the surface S, in the second case, $\mathbf{p}$ is called a **boundary point** of S.

The set of all boundary points of the surface S is called the **boundary** of S and denoted ∂S.

A smooth surface S is called **complete** if for any convergent sequence of points from S, the limit belongs to S as well. A complete surface with no boundary points is called **closed**.

Parametrized surfaces

Definition. Let $D \subset \mathbb{R}^2$ be a connected, bounded region. A continuous one-to-one map $\mathbf{X} : D \to \mathbb{R}^3$ is called a **parametrized surface**. The image $\mathbf{X}(D)$ is called the **underlying surface**.

The parametrized surface is **smooth** if $\mathbf{X}$ is smooth and, moreover, the vectors $\frac{\partial \mathbf{X}}{\partial s}(s_0, t_0)$ and $\frac{\partial \mathbf{X}}{\partial t}(s_0, t_0)$ are linearly independent for all $(s_0, t_0) \in D$. If this is the case, then the plane in $\mathbb{R}^3$ through the point $\mathbf{X}(s_0, t_0)$ parallel to vectors $\frac{\partial \mathbf{X}}{\partial s}(s_0, t_0)$ and $\frac{\partial \mathbf{X}}{\partial t}(s_0, t_0)$ is called the **tangent plane** to $\mathbf{X}(D)$ at $\mathbf{X}(s_0, t_0)$.

Example. Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function and consider a **level set** $P = \{(x, y, z) : f(x, y, z) = c\}$, $c \in \mathbb{R}$. If $\nabla f \neq \mathbf{0}$ at some point $p \in P$, then near that point P is the underlying surface of a parametrized surface. Moreover, the gradient $(\nabla f)(p)$ is orthogonal to the tangent plane at p.

Plane in space

Consider a map $\mathbf{X} : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\mathbf{X}\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix}.$$

Partial derivatives $\frac{\partial \mathbf{X}}{\partial s}$ and $\frac{\partial \mathbf{X}}{\partial t}$ are constant, namely, they are columns of the matrix $A = (a_{ij})$. Assume that the columns are linearly independent. Then $\mathbf{X}$ is a parametrized surface. The underlying surface is a plane Π. The tangent plane at every point is Π itself.

For a measurable set $D \subset \mathbb{R}^2$, the image $\mathbf{X}(D)$ is measurable in the plane Π. Moreover, $\mathrm{area}\big(\mathbf{X}(D)\big) = \alpha\, \mathrm{area}(D)$ for some fixed scalar α. To determine α, consider the unit square $Q = [0,1] \times [0,1]$. The image $\mathbf{X}(Q)$ is a parallelogram with adjacent sides represented by vectors $\frac{\partial \mathbf{X}}{\partial s}$ and $\frac{\partial \mathbf{X}}{\partial t}$. We obtain that $\alpha = \mathrm{area}\big(\mathbf{X}(Q)\big) = \|\frac{\partial \mathbf{X}}{\partial s} \times \frac{\partial \mathbf{X}}{\partial t}\|$.

Area of a surface

Let P be a smooth surface parametrized by $\mathbf{X} : D \to \mathbb{R}^3$. Then the area of P is

$$\operatorname{area}(P) = \iint_D \left\| \frac{\partial \mathbf{X}}{\partial s} \times \frac{\partial \mathbf{X}}{\partial t} \right\| ds\, dt.$$

Suppose P is the graph of a smooth function $g : D \to \mathbb{R}$, i.e., P is given by $z = g(x, y)$. We have a natural parametrization $\mathbf{X} : D \to \mathbb{R}^3$, $\mathbf{X}(x, y) = (x, y, g(x, y))$. Then $\frac{\partial \mathbf{X}}{\partial x} = (1, 0, g'_x)$ and $\frac{\partial \mathbf{X}}{\partial y} = (0, 1, g'_y)$. Consequently,

$$\frac{\partial \mathbf{X}}{\partial x} \times \frac{\partial \mathbf{X}}{\partial y} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 1 & 0 & g'_x \\ 0 & 1 & g'_y \end{vmatrix} = (-g'_x, -g'_y, 1).$$

It follows that

$$\operatorname{area}(P) = \iint_D \sqrt{1 + |g'_x|^2 + |g'_y|^2}\, dx\, dy.$$

Scalar surface integral

Scalar surface integral is an integral of a scalar function f over a parametrized surface $\mathbf{X} : D \to \mathbb{R}^3$ relative to the area element of the surface. It can be defined as a limit of Riemann sums

$$\mathcal{S}(f, R, \tau_j) = \sum_{j=1}^{k} f\big(\mathbf{x}(\tau_j)\big)\ \mathrm{area}\big(\mathbf{X}(D_j)\big),$$

where $R = \{D_1, D_2, \dots, D_k\}$ is a partition of D into small pieces and $\tau_j \in D_j$ for $1 \le j \le k$.

Theorem Let $\mathbf{X} : D \to \mathbb{R}^3$ be a smooth parametrized surface, where $D \subset \mathbb{R}^2$ is a bounded region. Then for any continuous function $f : \mathbf{X}(D) \to \mathbb{R}$,

$$\iint_{\mathbf{X}} f\, dS = \iint_{D} f\big(\mathbf{X}(s,t)\big) \left\| \frac{\partial \mathbf{X}}{\partial s} \times \frac{\partial \mathbf{X}}{\partial t} \right\| ds\, dt.$$

Vector surface integral

Vector surface integral is an integral of a vector field over a smooth parametrized surface. It is a scalar.

Definition. Let $\mathbf{X} : D \to \mathbb{R}^3$ be a smooth parametrized surface, where $D \subset \mathbb{R}^2$ is a bounded region. Then for any continuous vector field $\mathbf{F} : \mathbf{X}(D) \to \mathbb{R}^3$, the vector integral of $\mathbf{F}$ along $\mathbf{X}$ is

$$\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}\big(\mathbf{X}(s,t)\big) \cdot \mathbf{N}(s,t)\, ds\, dt,$$

where $\mathbf{N} = \frac{\partial \mathbf{X}}{\partial s} \times \frac{\partial \mathbf{X}}{\partial t}$, a normal vector to the surface.

Equivalently, $$\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} = \iint_D \begin{vmatrix} F_1 & F_2 & F_3 \\ \frac{\partial X_1}{\partial s} & \frac{\partial X_2}{\partial s} & \frac{\partial X_3}{\partial s} \\ \frac{\partial X_1}{\partial t} & \frac{\partial X_2}{\partial t} & \frac{\partial X_3}{\partial t} \end{vmatrix} ds\, dt.$$

Applications of surface integrals

- Mass of a shell

If f is the density of a shell P, then $\iint_P f\, dS$ is the mass of P.

- Center of mass of a shell

If f is the density of a shell P, then

$$\frac{\iint_P xf(x,y,z)\, dS}{\iint_P f\, dS}, \quad \frac{\iint_P yf(x,y,z)\, dS}{\iint_P f\, dS}, \quad \frac{\iint_P zf(x,y,z)\, dS}{\iint_P f\, dS}$$

are coordinates of the center of mass of P.

- Flux of fluid

If $\mathbf{F}$ is the velocity field of a fluid, then $\iint_P \mathbf{F} \cdot d\mathbf{S}$ is the flux of the fluid across the surface P.

Surface integrals and reparametrization

Given two smooth parametrized surfaces $\mathbf{X} : D_1 \to \mathbb{R}^3$ and $\mathbf{Y} : D_2 \to \mathbb{R}^3$, we say that $\mathbf{Y}$ is a **smooth reparametrization** of $\mathbf{X}$ if there exists an invertible function $\mathbf{H} : D_2 \to D_1$ such that $\mathbf{Y} = \mathbf{X} \circ \mathbf{H}$ and both $\mathbf{H}$ and $\mathbf{H}^{-1}$ are smooth.

Theorem Any scalar surface integral is invariant under smooth reparametrizations.

As a consequence, we can define the scalar integral of a function over a non-parametrized smooth surface.

Any vector surface integral can be represented as a scalar surface integral:

$$\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}\big(\mathbf{X}(s,t)\big) \cdot \mathbf{N}(s,t)\, ds\, dt = \iint_D (\mathbf{F} \cdot \mathbf{n})\, dS,$$

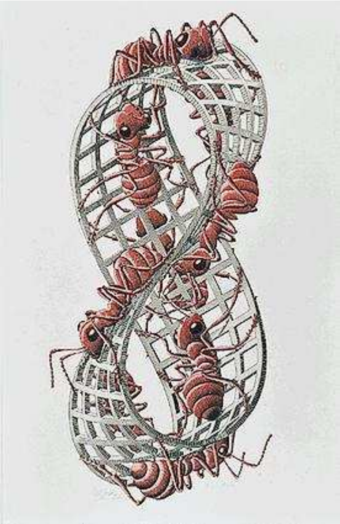
where $\mathbf{n} = \frac{\mathbf{N}}{\|\mathbf{N}\|}$ is a unit normal vector to the surface. Note that $\mathbf{n}$ depends continuously on a point on the surface, hence determining an **orientation** of $\mathbf{X}$.

A smooth reparametrization may be orientation-preserving (when $\mathbf{n}$ is preserved) or orientation-reversing (when $\mathbf{n}$ is changed to $-\mathbf{n}$).

Theorem Any vector surface integral is invariant under smooth orientation-preserving reparametrizations and changes its sign under orientation-reversing reparametrizations.

As a consequence, we can define the vector integral of a vector field over a non-parametrized, oriented smooth surface.

Moebius strip: non-orientable surface



M. C. Escher, 1963

Example

Let C denote the closed cylinder with bottom given by $z = 0$, top given by $z = 4$, and lateral surface given by $x^2 + y^2 = 9$. We orient C with outward normals.

$$\iint_C (x\mathbf{e}_1 + y\mathbf{e}_2) \cdot d\mathbf{S} = ?$$

The top of the cylinder is parametrized by $\mathbf{X}_{\mathrm{top}} : D \to \mathbb{R}^3$, $\mathbf{X}_{\mathrm{top}}(x, y) = (x, y, 4)$, where

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9\}.$$

The bottom is parametrized by $\mathbf{X}_{\mathrm{bot}} : D \to \mathbb{R}^3$, $\mathbf{X}_{\mathrm{bot}}(x, y) = (x, y, 0)$. The lateral surface is parametrized by $\mathbf{X}_{\mathrm{lat}} : [0, 2\pi] \times [0, 4] \to \mathbb{R}^3$, $\mathbf{X}_{\mathrm{lat}}(\phi, z) = (3\cos\phi, 3\sin\phi, z)$.

We have $\frac{\partial \mathbf{X}_{\text{top}}}{\partial x} = (1, 0, 0)$, $\frac{\partial \mathbf{X}_{\text{top}}}{\partial y} = (0, 1, 0)$. Hence $\frac{\partial \mathbf{X}_{\text{top}}}{\partial x} \times \frac{\partial \mathbf{X}_{\text{top}}}{\partial y} = \mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$.

Since $\mathbf{X}_{\text{bot}} = \mathbf{X}_{\text{top}} - (0, 0, 4)$, we also have $\frac{\partial \mathbf{X}_{\text{bot}}}{\partial x} = \mathbf{e}_1$, $\frac{\partial \mathbf{X}_{\text{bot}}}{\partial y} = \mathbf{e}_2$, and $\frac{\partial \mathbf{X}_{\text{bot}}}{\partial x} \times \frac{\partial \mathbf{X}_{\text{bot}}}{\partial y} = \mathbf{e}_3$.

Further, $\frac{\partial \mathbf{X}_{\text{lat}}}{\partial \phi} = (-3\sin\phi, 3\cos\phi, 0)$ and $\frac{\partial \mathbf{X}_{\text{lat}}}{\partial z} = (0, 0, 1)$. Therefore

$$\frac{\partial \mathbf{X}_{\text{lat}}}{\partial \phi} \times \frac{\partial \mathbf{X}_{\text{lat}}}{\partial z} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ -3\sin\phi & 3\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} =$$

$(3\cos\phi, 3\sin\phi, 0)$.

We observe that $\mathbf{X}_{\text{top}}$ and $\mathbf{X}_{\text{lat}}$ agree with the orientation of the surface C while $\mathbf{X}_{\text{bot}}$ does not. It follows that

$$\iint_C \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathbf{X}_{\text{top}}} \mathbf{F} \cdot d\mathbf{S} - \iint_{\mathbf{X}_{\text{bot}}} \mathbf{F} \cdot d\mathbf{S} + \iint_{\mathbf{X}_{\text{lat}}} \mathbf{F} \cdot d\mathbf{S}.$$

Integrating the vector field $\mathbf{F} = x\mathbf{e}_1 + y\mathbf{e}_2$ over each part of C, we obtain:

$$\iint_{\mathbf{X}_{\mathrm{top}}} \mathbf{F} \cdot d\mathbf{S} = \iint_D (x, y, 0) \cdot (0, 0, 1)\, dx\, dy = \iint_D 0\, dx\, dy = 0,$$

$$\iint_{\mathbf{X}_{\mathrm{bot}}} \mathbf{F} \cdot d\mathbf{S} = \iint_D (x, y, 0) \cdot (0, 0, 1)\, dx\, dy = \iint_D 0\, dx\, dy = 0,$$

$$\iint_{\mathbf{X}_{\mathrm{lat}}} \mathbf{F} \cdot d\mathbf{S} =$$

$$= \iint_{[0,2\pi]\times[0,4]} (3\cos\phi, 3\sin\phi, 0) \cdot (3\cos\phi, 3\sin\phi, 0)\, d\phi\, dz$$

$$= \iint_{[0,2\pi]\times[0,4]} 9\, d\phi\, dz = 72\pi.$$

Thus $\iint_C \mathbf{F} \cdot d\mathbf{S} = 72\pi.$