MATH 311

Topics in Applied Mathematics I

Lecture 1:
Systems of linear equations.

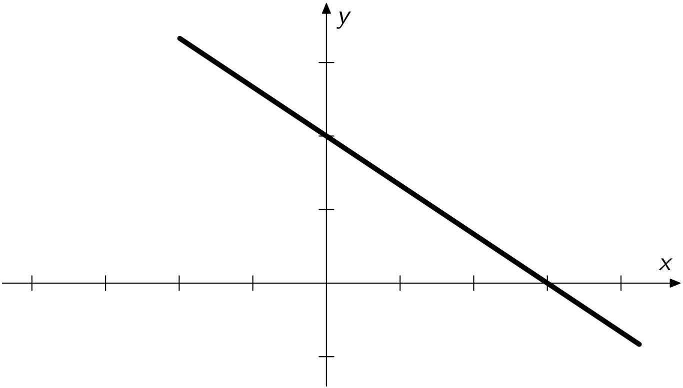
Linear equation

An equation $\boxed{2x + 3y = 6}$ is called *linear* because its solution set is a straight line in $\mathbb{R}^2$.

A *solution* of the equation is a pair of numbers $(\alpha, \beta) \in \mathbb{R}^2$ such that $2\alpha + 3\beta = 6$.

For example, $(3, 0)$ and $(0, 2)$ are solutions.

Alternatively, we can write the first solution as $x = 3,\ y = 0$.



$$2x + 3y = 6$$

General equation of a line: $\boxed{ax + by = c,}$
where x, y are variables and a, b, c are constants (except for the case $a = b = 0$).

Definition. A *linear equation* in variables $x_1, x_2, \dots, x_n$ is an equation of the form

$$\boxed{a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,}$$

where $a_1, \dots, a_n,$ and b are constants.

A *solution* of the equation is an array of numbers $(\gamma_1, \gamma_2, \dots, \gamma_n) \in \mathbb{R}^n$ such that

$$a_1\gamma_1 + a_2\gamma_2 + \cdots + a_n\gamma_n = b.$$

System of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Here $x_1, x_2, \ldots, x_n$ are variables and a_{ij}, b_j are constants.

A *solution* of the system is a common solution of all equations in the system.

Plenty of problems in mathematics and real world require solving systems of linear equations.

Problem Find the point of intersection of the lines $x - y = -2$ and $2x + 3y = 6$ in $\mathbb{R}^2$.

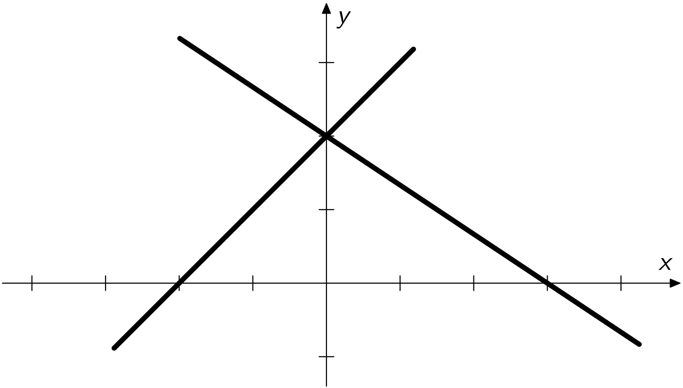
$$\begin{cases} x - y = -2 \\ 2x + 3y = 6 \end{cases} \iff \begin{cases} x = y - 2 \\ 2x + 3y = 6 \end{cases} \iff$$

$$\begin{cases} x = y - 2 \\ 2(y-2) + 3y = 6 \end{cases} \iff \begin{cases} x = y - 2 \\ 5y = 10 \end{cases} \iff$$

$$\begin{cases} x = y - 2 \\ y = 2 \end{cases} \iff \begin{cases} x = 0 \\ y = 2 \end{cases}$$

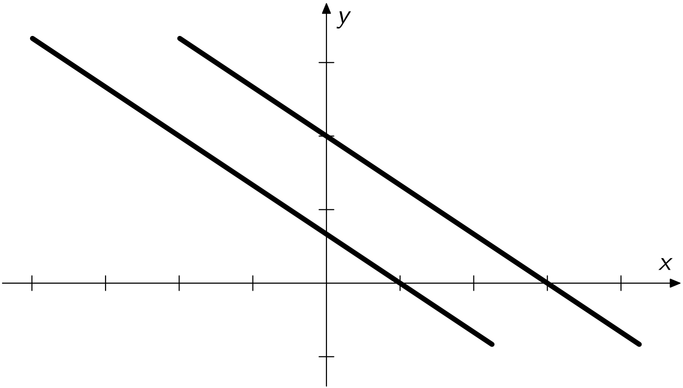
Solution: the lines intersect at the point $(0, 2)$.

Remark. The symbol of equivalence $\iff$ means that two systems have the same solutions.



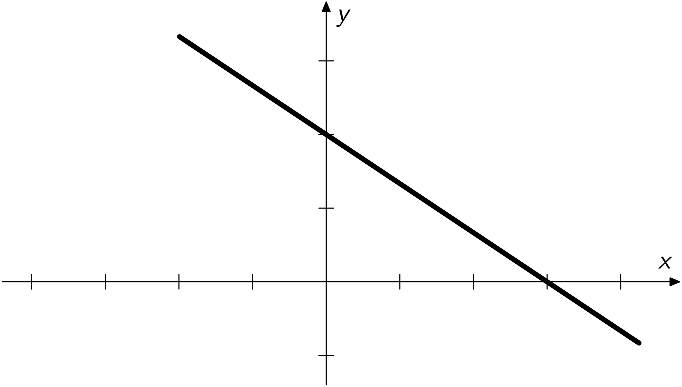
$$\begin{cases} x - y = -2 \\ 2x + 3y = 6 \end{cases}$$

$$x = 0,\ y = 2$$



$$\begin{cases} 2x + 3y = 2 \\ 2x + 3y = 6 \end{cases}$$

inconsistent system
(no solutions)



$$\begin{cases} 4x + 6y = 12 \\ 2x + 3y = 6 \end{cases} \iff 2x + 3y = 6$$

Solving systems of linear equations

Elimination method always works for systems of linear equations.

Algorithm: (1) pick a variable, solve one of the equations for it, and eliminate the variable from the other equations; (2) put aside the equation used in the elimination, and return to step (1).

The algorithm reduces the number of variables (as well as the number of equations), hence it stops after a finite number of steps.

After the algorithm stops, the system is simplified so that it should be clear how to complete solution.