

MATH 311

Topics in Applied Mathematics I

**Lecture 22:**  
**Eigenvalues and eigenvectors.**  
**Characteristic equation.**

## Eigenvalues and eigenvectors of a matrix

*Definition.* Let  $A$  be an  $n \times n$  matrix. A number  $\lambda \in \mathbb{R}$  is called an **eigenvalue** of the matrix  $A$  if  $A\mathbf{v} = \lambda\mathbf{v}$  for a nonzero column vector  $\mathbf{v} \in \mathbb{R}^n$ .

The vector  $\mathbf{v}$  is called an **eigenvector** of  $A$  belonging to (or associated with) the eigenvalue  $\lambda$ .

*Remarks.* • Alternative notation:  
eigenvalue = **characteristic value**,  
eigenvector = **characteristic vector**.

• The zero vector is never considered an eigenvector.

*Example.*  $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ .

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ -6 \end{pmatrix} = 3 \begin{pmatrix} 0 \\ -2 \end{pmatrix}.$$

Hence  $(1, 0)$  is an eigenvector of  $A$  belonging to the eigenvalue 2, while  $(0, -2)$  is an eigenvector of  $A$  belonging to the eigenvalue 3.

*Example.*  $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ .

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Hence  $(1, 1)$  is an eigenvector of  $A$  belonging to the eigenvalue 1, while  $(1, -1)$  is an eigenvector of  $A$  belonging to the eigenvalue  $-1$ .

Vectors  $\mathbf{v}_1 = (1, 1)$  and  $\mathbf{v}_2 = (1, -1)$  form a basis for  $\mathbb{R}^2$ . Consider a linear operator  $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by  $L(\mathbf{x}) = A\mathbf{x}$ . The matrix of  $L$  with respect to the basis  $\mathbf{v}_1, \mathbf{v}_2$  is  $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ .

Let  $A$  be an  $n \times n$  matrix. Consider a linear operator  $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$  given by  $L(\mathbf{x}) = A\mathbf{x}$ .

Let  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  be a nonstandard basis for  $\mathbb{R}^n$  and  $B$  be the matrix of the operator  $L$  with respect to this basis.

**Theorem** The matrix  $B$  is diagonal if and only if vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  are eigenvectors of  $A$ .

If this is the case, then the diagonal entries of the matrix  $B$  are the corresponding eigenvalues of  $A$ .

$$A\mathbf{v}_i = \lambda_i\mathbf{v}_i \iff B = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

## Eigenspaces

Let  $A$  be an  $n \times n$  matrix. Let  $\mathbf{v}$  be an eigenvector of  $A$  belonging to an eigenvalue  $\lambda$ .

Then  $A\mathbf{v} = \lambda\mathbf{v} \implies A\mathbf{v} = (\lambda I)\mathbf{v} \implies (A - \lambda I)\mathbf{v} = \mathbf{0}$ .

Hence  $\mathbf{v} \in N(A - \lambda I)$ , the nullspace of the matrix  $A - \lambda I$ .

Conversely, if  $\mathbf{x} \in N(A - \lambda I)$  then  $A\mathbf{x} = \lambda\mathbf{x}$ .

Thus the eigenvectors of  $A$  belonging to the eigenvalue  $\lambda$  are nonzero vectors from  $N(A - \lambda I)$ .

*Definition.* If  $N(A - \lambda I) \neq \{\mathbf{0}\}$  then it is called the **eigenspace** of the matrix  $A$  corresponding to the eigenvalue  $\lambda$ .

## How to find eigenvalues and eigenvectors?

**Theorem** Given a square matrix  $A$  and a scalar  $\lambda$ , the following statements are equivalent:

- $\lambda$  is an eigenvalue of  $A$ ,
- $N(A - \lambda I) \neq \{\mathbf{0}\}$ ,
- the matrix  $A - \lambda I$  is singular,
- $\det(A - \lambda I) = 0$ .

*Definition.*  $\det(A - \lambda I) = 0$  is called the **characteristic equation** of the matrix  $A$ .

Eigenvalues  $\lambda$  of  $A$  are roots of the characteristic equation. Associated eigenvectors of  $A$  are nonzero solutions of the equation  $(A - \lambda I)\mathbf{x} = \mathbf{0}$ .

*Example.*  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} \\ &= (a - \lambda)(d - \lambda) - bc \\ &= \lambda^2 - (a + d)\lambda + (ad - bc). \end{aligned}$$



Example.  $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$

$$\det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix}$$
$$= -\lambda^3 + c_1\lambda^2 - c_2\lambda + c_3,$$

where  $c_1 = a_{11} + a_{22} + a_{33}$  (the *trace* of  $A$ ),

$$c_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix},$$

$$c_3 = \det A.$$

**Theorem.** Let  $A = (a_{ij})$  be an  $n \times n$  matrix.

Then  $\det(A - \lambda I)$  is a polynomial of  $\lambda$  of degree  $n$ :

$$\det(A - \lambda I) = (-1)^n \lambda^n + c_1 \lambda^{n-1} + \cdots + c_{n-1} \lambda + c_n.$$

Furthermore,  $(-1)^{n-1} c_1 = a_{11} + a_{22} + \cdots + a_{nn}$   
and  $c_n = \det A$ .

*Definition.* The polynomial  $p(\lambda) = \det(A - \lambda I)$  is called the **characteristic polynomial** of the matrix  $A$ .

**Corollary** Any  $n \times n$  matrix has at most  $n$  eigenvalues.

*Example.*  $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ .

Characteristic equation:  $\begin{vmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{vmatrix} = 0$ .

$$(2 - \lambda)^2 - 1 = 0 \implies \lambda_1 = 1, \lambda_2 = 3.$$

$$(A - I)\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff x + y = 0.$$

The general solution is  $(-t, t) = t(-1, 1)$ ,  $t \in \mathbb{R}$ .

Thus  $\mathbf{v}_1 = (-1, 1)$  is an eigenvector associated with the eigenvalue 1. The corresponding eigenspace is the line spanned by  $\mathbf{v}_1$ .

$$(A - 3I)\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\iff \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff x - y = 0.$$

The general solution is  $(t, t) = t(1, 1)$ ,  $t \in \mathbb{R}$ .

Thus  $\mathbf{v}_2 = (1, 1)$  is an eigenvector associated with the eigenvalue 3. The corresponding eigenspace is the line spanned by  $\mathbf{v}_2$ .

*Summary.*  $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ .

- The matrix  $A$  has two eigenvalues: 1 and 3.
- The eigenspace of  $A$  associated with the eigenvalue 1 is the line  $t(-1, 1)$ .
- The eigenspace of  $A$  associated with the eigenvalue 3 is the line  $t(1, 1)$ .
- Eigenvectors  $\mathbf{v}_1 = (-1, 1)$  and  $\mathbf{v}_2 = (1, 1)$  of the matrix  $A$  form a basis for  $\mathbb{R}^2$ .
- Geometrically, the mapping  $\mathbf{x} \mapsto A\mathbf{x}$  is a stretch by a factor of 3 away from the line  $x + y = 0$  in the orthogonal direction.

*Example.*  $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$

Characteristic equation:

$$\begin{vmatrix} 1 - \lambda & 1 & -1 \\ 1 & 1 - \lambda & 1 \\ 0 & 0 & 2 - \lambda \end{vmatrix} = 0.$$

Expand the determinant by the 3rd row:

$$(2 - \lambda) \begin{vmatrix} 1 - \lambda & 1 \\ 1 & 1 - \lambda \end{vmatrix} = 0.$$

$$((1 - \lambda)^2 - 1)(2 - \lambda) = 0 \iff -\lambda(2 - \lambda)^2 = 0$$

$$\implies \lambda_1 = 0, \lambda_2 = 2.$$

$$A\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Convert the matrix to reduced row echelon form:

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A\mathbf{x} = \mathbf{0} \iff \begin{cases} x + y = 0, \\ z = 0. \end{cases}$$

The general solution is  $(-t, t, 0) = t(-1, 1, 0)$ ,  $t \in \mathbb{R}$ . Thus  $\mathbf{v}_1 = (-1, 1, 0)$  is an eigenvector associated with the eigenvalue 0. The corresponding eigenspace is the line spanned by  $\mathbf{v}_1$ .

$$(A - 2I)\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff x - y + z = 0.$$

The general solution is  $x = t - s$ ,  $y = t$ ,  $z = s$ , where  $t, s \in \mathbb{R}$ . Equivalently,

$$\mathbf{x} = (t - s, t, s) = t(1, 1, 0) + s(-1, 0, 1).$$

Thus  $\mathbf{v}_2 = (1, 1, 0)$  and  $\mathbf{v}_3 = (-1, 0, 1)$  are eigenvectors associated with the eigenvalue 2.

The corresponding eigenspace is the plane spanned by  $\mathbf{v}_2$  and  $\mathbf{v}_3$ .



Summary.  $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ .

- The matrix  $A$  has two eigenvalues: 0 and 2.
- The eigenvalue 0 is *simple*: the corresponding eigenspace is a line.
- The eigenvalue 2 is of *multiplicity* 2: the corresponding eigenspace is a plane.
- Eigenvectors  $\mathbf{v}_1 = (-1, 1, 0)$ ,  $\mathbf{v}_2 = (1, 1, 0)$ , and  $\mathbf{v}_3 = (-1, 0, 1)$  of the matrix  $A$  form a basis for  $\mathbb{R}^3$ .
- Geometrically, the map  $\mathbf{x} \mapsto A\mathbf{x}$  is the projection on the plane  $\text{Span}(\mathbf{v}_2, \mathbf{v}_3)$  along the lines parallel to  $\mathbf{v}_1$  with the subsequent scaling by a factor of 2.