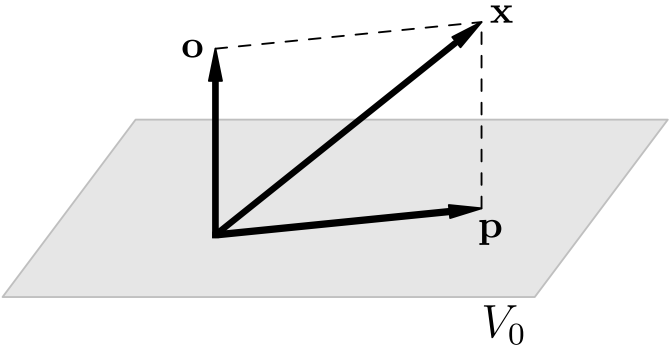
MATH 311

Topics in Applied Mathematics I

Lecture 30a:
The Gram-Schmidt process.

Orthogonal projection



Orthogonal projection

Theorem Let V be an inner product space and V_0 be a finite-dimensional subspace of V. Then any vector $\mathbf{x} \in V$ is uniquely represented as $\mathbf{x} = \mathbf{p} + \mathbf{o}$, where $\mathbf{p} \in V_0$ and $\mathbf{o} \perp V_0$.

The component $\mathbf{p}$ is the **orthogonal projection** of the vector $\mathbf{x}$ onto the subspace V_0. The distance from $\mathbf{x}$ to the subspace V_0 equals $\|\mathbf{o}\|$.

If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is an orthogonal basis for V_0 then

$$\mathbf{p} = \frac{\langle \mathbf{x}, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 + \frac{\langle \mathbf{x}, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{x}, \mathbf{v}_n \rangle}{\langle \mathbf{v}_n, \mathbf{v}_n \rangle}\mathbf{v}_n.$$

The Gram-Schmidt orthogonalization process

Let V be a vector space with an inner product. Suppose $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ is a basis for V. Let

$$\mathbf{v}_1 = \mathbf{x}_1,$$

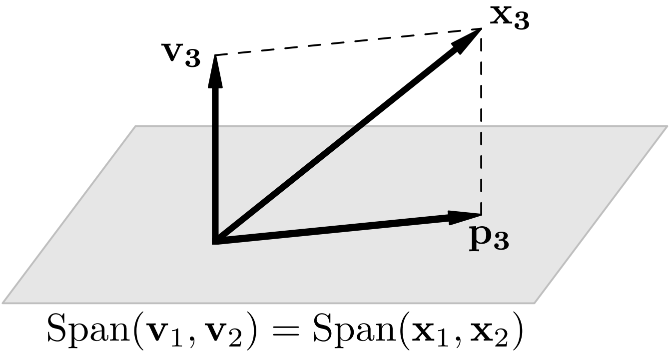
$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1,$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 - \frac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\mathbf{v}_n = \mathbf{x}_n - \frac{\langle \mathbf{x}_n, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 - \cdots - \frac{\langle \mathbf{x}_n, \mathbf{v}_{n-1} \rangle}{\langle \mathbf{v}_{n-1}, \mathbf{v}_{n-1} \rangle}\mathbf{v}_{n-1}.$$

Then $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ is an orthogonal basis for V.


$\mathbf{x_3}$
$\mathbf{v_3}$
$\mathbf{p_3}$
$\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2) = \mathrm{Span}(\mathbf{x}_1, \mathbf{x}_2)$

Any basis	$\longrightarrow$	Orthogonal basis
$\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$		$\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$

Properties of the Gram-Schmidt process:

- $\mathbf{v}_k = \mathbf{x}_k - (\alpha_1 \mathbf{x}_1 + \dots + \alpha_{k-1} \mathbf{x}_{k-1})$, $1 \le k \le n$;
- the span of $\mathbf{v}_1, \dots, \mathbf{v}_k$ is the same as the span of $\mathbf{x}_1, \dots, \mathbf{x}_k$;
- $\mathbf{v}_k$ is orthogonal to $\mathbf{x}_1, \dots, \mathbf{x}_{k-1}$;
- $\mathbf{v}_k = \mathbf{x}_k - \mathbf{p}_k$, where $\mathbf{p}_k$ is the orthogonal projection of the vector $\mathbf{x}_k$ on the subspace spanned by $\mathbf{x}_1, \dots, \mathbf{x}_{k-1}$;
- $\|\mathbf{v}_k\|$ is the distance from $\mathbf{x}_k$ to the subspace spanned by $\mathbf{x}_1, \dots, \mathbf{x}_{k-1}$.

Normalization

Let V be a vector space with an inner product.
Suppose $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ is an orthogonal basis for V.
Let $\mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}$, $\mathbf{w}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}$, $\dots$, $\mathbf{w}_n = \frac{\mathbf{v}_n}{\|\mathbf{v}_n\|}$.
Then $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n$ is an orthonormal basis for V.

Theorem Any finite-dimensional vector space with an inner product has an orthonormal basis.

Remark. An infinite-dimensional vector space with an inner product may or may not have an orthonormal basis.

Orthogonalization / Normalization

An alternative form of the Gram-Schmidt process combines orthogonalization with normalization.

Suppose $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ is a basis for an inner product space V. Let

$$\mathbf{v}_1 = \mathbf{x}_1, \quad \mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|},$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \langle \mathbf{x}_2, \mathbf{w}_1 \rangle \mathbf{w}_1, \quad \mathbf{w}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|},$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \langle \mathbf{x}_3, \mathbf{w}_1 \rangle \mathbf{w}_1 - \langle \mathbf{x}_3, \mathbf{w}_2 \rangle \mathbf{w}_2, \quad \mathbf{w}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\mathbf{v}_n = \mathbf{x}_n - \langle \mathbf{x}_n, \mathbf{w}_1 \rangle \mathbf{w}_1 - \cdots - \langle \mathbf{x}_n, \mathbf{w}_{n-1} \rangle \mathbf{w}_{n-1},$$

$$\mathbf{w}_n = \frac{\mathbf{v}_n}{\|\mathbf{v}_n\|}.$$

Then $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$ is an orthonormal basis for V.

Problem. Let Π be the plane spanned by vectors $\mathbf{x}_1 = (1,1,0)$ and $\mathbf{x}_2 = (0,1,1)$.

(i) Find the orthogonal projection of the vector $\mathbf{y} = (4,0,-1)$ onto the plane Π.

(ii) Find the distance from $\mathbf{y}$ to Π.

First we apply the Gram-Schmidt process to the basis $\mathbf{x}_1, \mathbf{x}_2$:

$$\mathbf{v}_1 = \mathbf{x}_1 = (1,1,0),$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 = (0,1,1) - \frac{1}{2}(1,1,0) = (-1/2, 1/2, 1).$$

Now that $\mathbf{v}_1, \mathbf{v}_2$ is an orthogonal basis for Π, the orthogonal projection of $\mathbf{y}$ onto Π is

$$\mathbf{p} = \frac{\langle \mathbf{y}, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 + \frac{\langle \mathbf{y}, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2 = \frac{4}{2}(1,1,0) + \frac{-3}{3/2}(-1/2, 1/2, 1)$$

$$= (2,2,0) + (1,-1,-2) = (3,1,-2).$$

The distance from $\mathbf{y}$ to Π is $\|\mathbf{y} - \mathbf{p}\| = \|(1,-1,1)\| = \sqrt{3}.$

Problem. Approximate the function $f(x) = e^x$ on the interval $[-1, 1]$ by a quadratic polynomial.

The best approximation would be a polynomial $p(x)$ that minimizes the distance relative to the uniform norm:

$$\|f - p\|_\infty = \max_{|x| \le 1} |f(x) - p(x)|.$$

However there is no analytic way to find such a polynomial. Instead, one can find a **"least squares"** approximation that minimizes the integral norm

$$\|f - p\|_2 = \left(\int_{-1}^{1} |f(x) - p(x)|^2 \, dx \right)^{1/2}.$$

The norm $\|\cdot\|_2$ is induced by the inner product

$$\langle g, h\rangle = \int_{-1}^{1} g(x)h(x)\,dx.$$

Therefore $\|f - p\|_2$ is minimal if p is the orthogonal projection of the function f on the subspace $\mathcal{P}_3$ of quadratic polynomials.

We should apply the Gram-Schmidt process to the polynomials $1, x, x^2,$ which form a basis for $\mathcal{P}_3$. This would yield an orthogonal basis p_0, p_1, p_2. Then

$$p(x) = \frac{\langle f, p_0\rangle}{\langle p_0, p_0\rangle}p_0(x) + \frac{\langle f, p_1\rangle}{\langle p_1, p_1\rangle}p_1(x) + \frac{\langle f, p_2\rangle}{\langle p_2, p_2\rangle}p_2(x).$$