MATH 311
Topics in Applied Mathematics I

Lecture 37:
Surface integrals.
Gauss' theorem.
Stokes' theorem.

Scalar surface integral

Scalar surface integral is an integral of a scalar function f over a parametrized surface $\mathbf{X} : D \to \mathbb{R}^3$ relative to the area element of the surface. It can be defined as a limit of Riemann sums

$$\mathcal{S}(f, R, \tau_j) = \sum_{j=1}^{k} f\big(\mathbf{X}(\tau_j)\big)\ \mathrm{area}\big(\mathbf{X}(D_j)\big),$$

where $R = \{D_1, D_2, \ldots, D_k\}$ is a partition of D into small pieces and $\tau_j \in D_j$ for $1 \leq j \leq k$.

Theorem Let $\mathbf{X} : D \to \mathbb{R}^3$ be a smooth parametrized surface, where $D \subset \mathbb{R}^2$ is a bounded region. Then for any continuous function $f : \mathbf{X}(D) \to \mathbb{R}$,

$$\iint_{\mathbf{X}} f\, dS = \iint_{D} f\big(\mathbf{X}(s,t)\big) \left\| \frac{\partial \mathbf{X}}{\partial s} \times \frac{\partial \mathbf{X}}{\partial t} \right\| ds\, dt.$$

Vector surface integral

Vector surface integral is an integral of a vector field over a smooth parametrized surface. It is a scalar.

Definition. Let $\mathbf{X} : D \to \mathbb{R}^3$ be a smooth parametrized surface, where $D \subset \mathbb{R}^2$ is a bounded region. Then for any continuous vector field $\mathbf{F} : \mathbf{X}(D) \to \mathbb{R}^3$, the vector integral of $\mathbf{F}$ along $\mathbf{X}$ is

$$\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}\big(\mathbf{X}(s,t)\big) \cdot \mathbf{N}(s,t)\, ds\, dt,$$

where $\mathbf{N} = \frac{\partial \mathbf{X}}{\partial s} \times \frac{\partial \mathbf{X}}{\partial t}$, a normal vector to the surface.

Equivalently, $$\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} = \iint_D \begin{vmatrix} F_1 & F_2 & F_3 \\ \frac{\partial X_1}{\partial s} & \frac{\partial X_2}{\partial s} & \frac{\partial X_3}{\partial s} \\ \frac{\partial X_1}{\partial t} & \frac{\partial X_2}{\partial t} & \frac{\partial X_3}{\partial t} \end{vmatrix} ds\, dt.$$

Applications of surface integrals

- Mass of a shell

If f is the density of a shell P, then $\iint_P f\, dS$ is the mass of P.

- Center of mass of a shell

If f is the density of a shell P, then

$$\frac{\iint_P xf(x,y,z)\, dS}{\iint_P f\, dS},\quad \frac{\iint_P yf(x,y,z)\, dS}{\iint_P f\, dS},\quad \frac{\iint_P zf(x,y,z)\, dS}{\iint_P f\, dS}$$

are coordinates of the center of mass of P.

- Flux of fluid

If $\mathbf{F}$ is the velocity field of a fluid, then $\iint_P \mathbf{F} \cdot d\mathbf{S}$ is the flux of the fluid across the surface P.

Surface integrals and reparametrization

Given two smooth parametrized surfaces $\mathbf{X} : D_1 \to \mathbb{R}^3$ and $\mathbf{Y} : D_2 \to \mathbb{R}^3$, we say that $\mathbf{Y}$ is a **smooth reparametrization** of $\mathbf{X}$ if there exists an invertible function $\mathbf{H} : D_2 \to D_1$ such that $\mathbf{Y} = \mathbf{X} \circ \mathbf{H}$ and both $\mathbf{H}$ and $\mathbf{H}^{-1}$ are smooth.

Theorem Any scalar surface integral is invariant under smooth reparametrizations.

As a consequence, we can define the scalar integral of a function over a non-parametrized smooth surface.

Any vector surface integral can be represented as a scalar surface integral:

$$\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}\big(\mathbf{X}(s,t)\big) \cdot \mathbf{N}(s,t)\, ds\, dt = \iint_D (\mathbf{F} \cdot \mathbf{n})\, dS,$$

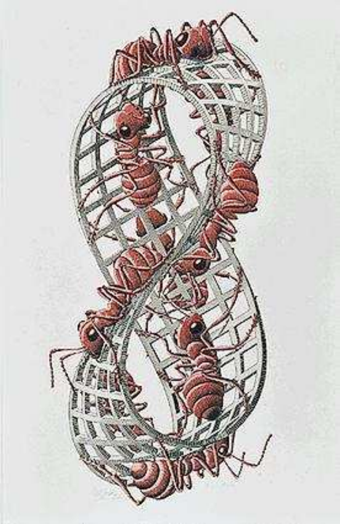
where $\mathbf{n} = \frac{\mathbf{N}}{\|\mathbf{N}\|}$ is a unit normal vector to the surface. Note that $\mathbf{n}$ depends continuously on a point on the surface, hence determining an **orientation** of $\mathbf{X}$.

A smooth reparametrization may be orientation-preserving (when $\mathbf{n}$ is preserved) or orientation-reversing (when $\mathbf{n}$ is changed to $-\mathbf{n}$).

Theorem Any vector surface integral is invariant under smooth orientation-preserving reparametrizations and changes its sign under orientation-reversing reparametrizations.

As a consequence, we can define the vector integral of a vector field over a non-parametrized, oriented smooth surface.

Moebius strip: non-orientable surface



M. C. Escher, 1963

Problem. Let C denote the closed cylinder with bottom given by $z = 0$, top given by $z = 4$, and lateral surface given by $x^2 + y^2 = 9$. We orient ∂C with outward normals. Find the integral of a vector field $\mathbf{F}(x, y, z) = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ along ∂C.

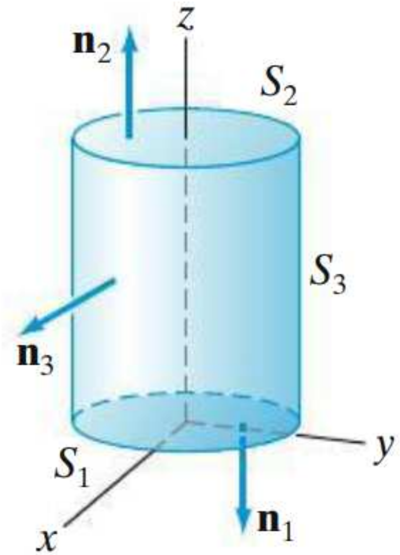
To evaluate the integral, we cut the boundary ∂C into three parts: the top, the bottom and the lateral surface.

The top of the cylinder is parametrized by $\mathbf{X}_{\rm top} : D \to \mathbb{R}^3$, $\mathbf{X}_{\rm top}(x, y) = (x, y, 4)$, where

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9\}.$$

The bottom is parametrized by $\mathbf{X}_{\rm bot} : D \to \mathbb{R}^3$, $\mathbf{X}_{\rm bot}(x, y) = (x, y, 0)$.

The lateral surface is parametrized by $\mathbf{X}_{\rm lat} : [0, 2\pi] \times [0, 4] \to \mathbb{R}^3$, $\mathbf{X}_{\rm lat}(\phi, z) = (3\cos\phi, 3\sin\phi, z)$.



z
$\mathbf{n}_2$
S_2
S_3
$\mathbf{n}_3$
y
S_1
x
$\mathbf{n}_1$

We have $\frac{\partial \mathbf{X}_{\rm top}}{\partial x} = (1,0,0)$, $\frac{\partial \mathbf{X}_{\rm top}}{\partial y} = (0,1,0)$. Hence $\frac{\partial \mathbf{X}_{\rm top}}{\partial x} \times \frac{\partial \mathbf{X}_{\rm top}}{\partial y} = \mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$.

Since $\mathbf{X}_{\rm bot} = \mathbf{X}_{\rm top} - (0,0,4)$, we also have $\frac{\partial \mathbf{X}_{\rm bot}}{\partial x} = \mathbf{e}_1$, $\frac{\partial \mathbf{X}_{\rm bot}}{\partial y} = \mathbf{e}_2$, and $\frac{\partial \mathbf{X}_{\rm bot}}{\partial x} \times \frac{\partial \mathbf{X}_{\rm bot}}{\partial y} = \mathbf{e}_3$.

Further, $\frac{\partial \mathbf{X}_{\rm lat}}{\partial \phi} = (-3\sin\phi, 3\cos\phi, 0)$ and $\frac{\partial \mathbf{X}_{\rm lat}}{\partial z} = (0,0,1)$. Therefore

$$\frac{\partial \mathbf{X}_{\rm lat}}{\partial \phi} \times \frac{\partial \mathbf{X}_{\rm lat}}{\partial z} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ -3\sin\phi & 3\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = (3\cos\phi, 3\sin\phi, 0).$$

We observe that $\mathbf{X}_{\rm top}$ and $\mathbf{X}_{\rm lat}$ agree with the orientation of the surface C while $\mathbf{X}_{\rm bot}$ does not. It follows that

$$\oiint_{\partial C} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathbf{X}_{\rm top}} \mathbf{F} \cdot d\mathbf{S} - \iint_{\mathbf{X}_{\rm bot}} \mathbf{F} \cdot d\mathbf{S} + \iint_{\mathbf{X}_{\rm lat}} \mathbf{F} \cdot d\mathbf{S}.$$

Integrating the vector field $\mathbf{F} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ along each part of the boundary of C, we obtain:

$$\iint_{\mathbf{X}_{\text{top}}} \mathbf{F} \cdot d\mathbf{S} = \iint_D (x, y, 4) \cdot (0, 0, 1)\, dx\, dy = \iint_D 4\, dx\, dy = 36\pi,$$

$$\iint_{\mathbf{X}_{\text{bot}}} \mathbf{F} \cdot d\mathbf{S} = \iint_D (x, y, 0) \cdot (0, 0, 1)\, dx\, dy = \iint_D 0\, dx\, dy = 0,$$

$$\iint_{\mathbf{X}_{\text{lat}}} \mathbf{F} \cdot d\mathbf{S} =$$

$$= \iint_{[0,2\pi]\times[0,4]} (3\cos\phi, 3\sin\phi, z) \cdot (3\cos\phi, 3\sin\phi, 0)\, d\phi\, dz$$

$$= \iint_{[0,2\pi]\times[0,4]} 9\, d\phi\, dz = 72\pi.$$

Thus $\displaystyle\oiint_{\partial C} \mathbf{F} \cdot d\mathbf{S} = 36\pi - 0 + 72\pi = 108\pi.$

Gauss's Theorem (a.k.a. Divergence Theorem in $\mathbb{R}^3$)

Theorem Let $D \subset \mathbb{R}^3$ be a closed, bounded region with piecewise smooth boundary ∂D (not necessarily connected) oriented by **outward** unit normals to D. Then for any smooth vector field $\mathbf{F}$ on D,

$$\oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{F} \, dV.$$

Corollary If a smooth vector field $\mathbf{F} : D \to \mathbb{R}^3$ has no divergence, $\nabla \cdot \mathbf{F} = 0$, then $\oiint_C \mathbf{F} \cdot d\mathbf{S} = 0$ for any closed, piecewise smooth surface C that bounds a subregion of D.

Problem. Let C denote the closed cylinder with bottom given by $z = 0$, top given by $z = 4$, and lateral surface given by $x^2 + y^2 = 9$. We orient ∂C with outward normals. Find the integral of a vector field $\mathbf{F}(x, y, z) = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ along ∂C.

Now let us use Gauss' Theorem:

$$\oiint_{\partial C} \mathbf{F} \cdot d\mathbf{S} = \iiint_C \nabla \cdot \mathbf{F} \, dV$$

$$= \iiint_C \left(\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) \right) dx \, dy \, dz$$

$$= \iiint_C 3 \, dx \, dy \, dz = 3 \, \mathrm{volume}(C) = 108\pi.$$

Stokes's Theorem

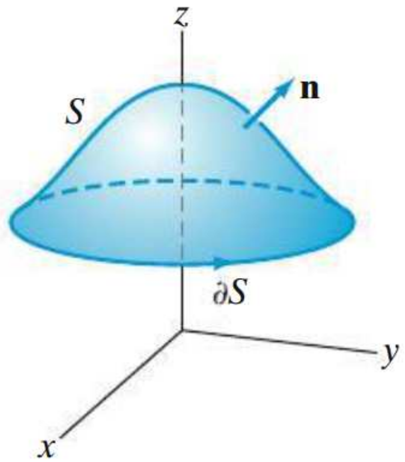
Suppose S is an oriented surface in $\mathbb{R}^3$ bounded by an oriented curve ∂S. We say that ∂S is **oriented consistently with** S if, as one traverses ∂S, the surface S is on the left when looking down from the tip of $\mathbf{n}$, the unit normal vector indicating the orientation of S.

Theorem Let $S \subset \mathbb{R}^3$ be a bounded, piecewise smooth oriented surface with piecewise smooth boundary ∂S oriented consistently with S. Then for any smooth vector field $\mathbf{F}$ on S,

$$\iint_S \mathrm{curl}(\mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s}.$$

Corollary If the surface S is closed (i.e., has no boundary), then for any smooth vector field $\mathbf{F}$ on S,

$$\iint_S \mathrm{curl}(\mathbf{F}) \cdot d\mathbf{S} = 0.$$


z
n
S
∂S
y
x

Example

Suppose that a bounded, piecewise smooth surface $S \subset \mathbb{R}^3$ is contained in the xy-coordinate plane, that is, $S = D \times \{0\}$ for a domain $D \subset \mathbb{R}^2$. We orient S by the upward unit normal vector $\mathbf{n} = (0,0,1)$ and orient the boundary $\partial S = \partial D \times \{0\}$ consistently with S. Further, suppose that $\mathbf{F}$ is a horizontal vector field, $\mathbf{F} = (M, N, 0)$. By Stokes' Theorem,

$$\iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s}.$$

Recall that $\iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = \iint_S \operatorname{curl}(\mathbf{F}) \cdot \mathbf{n}\, dS$. We obtain

$$\operatorname{curl}(\mathbf{F}) \cdot \mathbf{n} = \begin{vmatrix} 0 & 0 & 1 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.$$

It follows that this particular case of Stokes' Theorem is equivalent to Green's Theorem.