MATH 311

Topics in Applied Mathematics I

Lecture 21:
Review of differential calculus.
Differentiation in normed vector spaces.

Limit of a sequence

Definition. Sequence $x_1, x_2, x_3, \dots$ of real numbers is said to **converge** to a real number a if for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_n - a| < \varepsilon$ for all $n \geq N$. The number a is called the **limit** of $\{x_n\}$.

Notation: $\lim_{n\to\infty} x_n = a$, or $x_n \to a$ as $n \to \infty$.

Note that $d(x, y) = |x - y|$ is the distance between points x and y on the real line.

The condition $|x_n - a| < \varepsilon$ is equivalent to $x_n \in (a - \varepsilon, a + \varepsilon)$. The interval $(a - \varepsilon, a + \varepsilon)$ is called the ε-**neighborhood** of the point a. The convergence $x_n \to a$ means that any ε-neighborhood of a contains all but finitely many elements of the sequence $\{x_n\}$.

Limit of a function

Suppose $f : E \to \mathbb{R}$ is a function defined on a set $E \subset \mathbb{R}$.

Definition. We say that the function f **converges to a limit** $L \in \mathbb{R}$ at a point a if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$0 < |x - a| < \delta \text{ implies } |f(x) - L| < \varepsilon.$$

Notation: $L = \lim_{x \to a} f(x)$ or $f(x) \to L$ as $x \to a$.

Theorem Let I be an open interval containing a point $a \in \mathbb{R}$ and f be a function defined on $I \setminus \{a\}$. Then $f(x) \to L$ as $x \to a$ if and only if for any sequence $\{x_n\}$ of elements of $I \setminus \{a\}$,

$$\lim_{n \to \infty} x_n = a \quad \text{implies} \quad \lim_{n \to \infty} f(x_n) = L.$$

Continuity

Definition. Given a set $E \subset \mathbb{R}$, a function $f : E \to \mathbb{R}$, and a point $c \in E$, the function f is **continuous at** c if

$$f(c) = \lim_{x \to c} f(x).$$

That is, if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $|x - c| < \delta$ and $x \in E$ imply $|f(x) - f(c)| < \varepsilon$.

Theorem A function $f : E \to \mathbb{R}$ is continuous at a point $c \in E$ if and only if for any sequence $\{x_n\}$ of elements of E, $x_n \to c$ as $n \to \infty$ implies $f(x_n) \to f(c)$ as $n \to \infty$.

We say that the function f is **continuous on** a set $E_0 \subset E$ if f is continuous at every point $c \in E_0$. The function f is **continuous** if it is continuous on the entire domain E.

Topology of the real line

Definition. A sequence $\{x_n\}$ of real numbers is called a **Cauchy sequence** if for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_n - x_m| < \varepsilon$ whenever $n, m \geq N$.

Theorem (Cauchy) Any Cauchy sequence is convergent.

This property of $\mathbb{R}$ is called **completeness**.

Theorem (Bolzano-Weierstrass) Every bounded sequence of real numbers has a convergent subsequence.

This property of $\mathbb{R}$ is called **local compactness**.

A set $S \subset \mathbb{R}$ is called **compact** if any sequence of its elements has a subsequence converging to a limit in S. For example, any closed bounded interval $[a, b]$ is compact.

Extreme Value Theorem If $S \subset \mathbb{R}$ is compact, then any continuous function $f : S \to \mathbb{R}$ attains its extreme values on S.

The derivative

Definition. A real function f is said to be **differentiable** at a point $a \in \mathbb{R}$ if it is defined on an open interval containing a and the limit

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. The limit is denoted $f'(a)$ and called the **derivative** of f at a. An equivalent condition is

$$f(a+h) = f(a) + f'(a)h + r(h), \text{ where } \lim_{h \to 0} r(h)/h = 0.$$

If a function f is differentiable at a point a, then it is continuous at a.

Suppose that a function f is defined and differentiable on an interval I. Then the derivative of f can be regarded as a function on I. *Notation:* f', $\dot{f}$, $\dfrac{df}{dx}$, $D_x f$, $f^{(1)}$.

Differentiability theorems

Sum Rule If functions f and g are differentiable at a point $a \in \mathbb{R}$, then the sum $f + g$ is also differentiable at a. Moreover, $(f + g)'(a) = f'(a) + g'(a)$.

Homogeneous Rule If a function f is differentiable at a point $a \in \mathbb{R}$, then for any $r \in \mathbb{R}$ the scalar multiple rf is also differentiable at a. Moreover, $(rf)'(a) = rf'(a)$.

Difference Rule If functions f and g are differentiable at a point $a \in \mathbb{R}$, then the difference $f - g$ is also differentiable at a. Moreover, $(f - g)'(a) = f'(a) - g'(a)$.

Differentiability theorems

Product Rule If functions f and g are differentiable at a point $a \in \mathbb{R}$, then the product fg is also differentiable at a. Moreover, $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$.

Reciprocal Rule If a function f is differentiable at a point $a \in \mathbb{R}$ and $f(a) \neq 0$, then the function $1/f$ is also differentiable at a. Moreover, $(1/f)'(a) = -f'(a)/f^2(a)$.

Quotient Rule If functions f and g are differentiable at $a \in \mathbb{R}$ and $g(a) \neq 0$, then the quotient f/g is also differentiable at a. Moreover,

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}.$$

Differentiability theorems

Chain Rule If a function f is differentiable at a point $a \in \mathbb{R}$ and a function g is differentiable at $f(a)$, then the composition $g \circ f$ is differentiable at a. Moreover, $(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$.

Derivative of the inverse function Suppose f is an invertible continuous function. If f is differentiable at a point a and $f'(a) \neq 0$, then the inverse function is differentiable at the point $b = f(a)$ and, moreover,

$$(f^{-1})'(b) = \frac{1}{f'(a)}.$$

In the case $f'(a) = 0$, the inverse function f^{-1} is not differentiable at $f(a)$.

Properties of differentiable functions

Fermat's Theorem If a function f is differentiable at a point c of local extremum (maximum or minimum), then $f'(c) = 0$.

Rolle's Theorem If a function f is continuous on a closed interval $[a, b]$, differentiable on the open interval (a, b), and if $f(a) = f(b)$, then $f'(c) = 0$ for some $c \in (a, b)$.

Mean Value Theorem If a function f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that $f(b) - f(a) = f'(c)\,(b - a)$.

Problem. Find $\min_{x>0} x^x$.

The function $f(x) = x^x$ is well defined and positive on $(0, \infty)$. Hence

$$f(x) = e^{\log f(x)} = e^{\log x^x} = e^{x \log x}$$

for all $x > 0$. That is, $f(x) = g(h(x))$, where $h(x) = x \log x$ and $g(y) = e^y$. Using the Chain Rule and the Product Rule, we obtain

$$\begin{aligned} f'(x) &= e^{x \log x}(x \log x)' = x^x \Big((x)' \log x + x(\log x)' \Big) \\ &= x^x(\log x + 1). \end{aligned}$$

It follows that $f'(x) < 0$ for $0 < x < 1/e$ and $f'(x) > 0$ for $x > 1/e$. Hence the function f is strictly decreasing on $(0, 1/e]$ and strictly increasing on $[1/e, \infty)$. Therefore

$$\min_{x>0} f(x) = f(1/e) = (1/e)^{1/e} = e^{-1/e}.$$

Convergence in normed vector spaces

Suppose V is a vector space endowed with a norm $\|\cdot\|$. The norm gives rise to a distance function $\mathrm{dist}(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$.

Definition. We say that a sequence of vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots$ converges to a vector $\mathbf{u}$ in the normed vector space V if $\|\mathbf{v}_k - \mathbf{u}\| \to 0$ as $k \to \infty$.

In the case $V = \mathbb{R}^n$, a sequence of vectors converges with respect to a norm if and only if it converges in each coordinate. In the case $V = \mathcal{M}_{m,n}(\mathbb{R})$, a sequence of matrices converges with respect to a norm if and only if it converges in each entry.

Similarly, in the case $\dim V < \infty$ we can choose a finite basis $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n$. Any vector $\mathbf{v} \in V$ can be expanded into a linear combination $\mathbf{v} = x_1\mathbf{w}_1 + x_2\mathbf{w}_2 + \cdots + x_n\mathbf{w}_n$. Then a sequence of vectors converges with respect to a norm if and only if each of the coordinates x_i converges.

Vector-valued functions

Suppose V is a vector space endowed with a norm $\|\cdot\|$.

Definition. We say that a function $\mathbf{v} : X \to V$ defined on a set $X \subset \mathbb{R}$ converges to a limit $\mathbf{u} \in V$ at a point $a \in \mathbb{R}$ if $\|\mathbf{v}(x) - \mathbf{u}\| \to 0$ as $x \to a$.

Further, we say that the function $\mathbf{v}$ is continuous at a point $c \in X$ if $\mathbf{v}(c) = \lim_{x \to c} \mathbf{v}(x)$.

Finally, the function $\mathbf{v}$ is said to be differentiable at a point $a \in \mathbb{R}$ if it is defined on an open interval containing a and the limit

$$\lim_{h \to 0} \frac{1}{h}\big(f(a+h) - f(a)\big)$$

exists. The limit is denoted $\mathbf{v}'(a)$ and called the derivative of $\mathbf{v}$ at a.

Differentiability theorems

Sum Rule If functions $\mathbf{v} : X \to V$ and $\mathbf{w} : X \to V$ are differentiable at a point $a \in \mathbb{R}$, then the sum $\mathbf{v} + \mathbf{w}$ is also differentiable at a. Moreover, $(\mathbf{v} + \mathbf{w})'(a) = \mathbf{v}'(a) + \mathbf{w}'(a)$.

Homogeneous Rule If a function $\mathbf{v} : X \to V$ is differentiable at a point $a \in \mathbb{R}$, then for any $r \in \mathbb{R}$ the scalar multiple $r\mathbf{v}$ is also differentiable at a. Moreover, $(r\mathbf{v})'(a) = r\mathbf{v}'(a)$.

Difference Rule If functions $\mathbf{v} : X \to V$ and $\mathbf{w} : X \to V$ are differentiable at a point $a \in \mathbb{R}$, then the difference $\mathbf{v} - \mathbf{w}$ is also differentiable at a. Moreover, $(\mathbf{v} - \mathbf{w})'(a) = \mathbf{v}'(a) - \mathbf{w}'(a)$.

Differentiability theorems

Product Rule #1 If functions $f : X \to \mathbb{R}$ and $\mathbf{v} : X \to V$ are differentiable at a point $a \in \mathbb{R}$, then the scalar multiple $f\mathbf{v}$ is also differentiable at a. Moreover, $(f\mathbf{v})'(a) = f'(a)\mathbf{v}(a) + f(a)\mathbf{v}'(a)$.

Product Rule #2 Assume that the norm on V is induced by an inner product $\langle \cdot, \cdot \rangle$. If functions $\mathbf{v} : X \to V$ and $\mathbf{w} : X \to V$ are differentiable at a point $a \in \mathbb{R}$, then the inner product $\langle \mathbf{v}, \mathbf{w} \rangle$ is also differentiable at a. Moreover, $(\langle \mathbf{v}, \mathbf{w} \rangle)'(a) = \langle \mathbf{v}'(a), \mathbf{w}(a) \rangle + \langle \mathbf{v}(a), \mathbf{w}'(a) \rangle$.

Chain Rule If a function $f : X \to \mathbb{R}$ is differentiable at a point $a \in \mathbb{R}$ and a function $\mathbf{v} : Y \to V$ is differentiable at $f(a)$, then the composition $\mathbf{v} \circ f$ is differentiable at a. Moreover, $(\mathbf{v} \circ f)'(a) = f'(a)\mathbf{v}'(f(a))$.

Partial derivative

Consider a function $f : X \to V$ that is defined in a domain $X \subset \mathbb{R}^n$ and takes values in a normed vector space V. The function f depends on n real variables: $f = f(x_1, x_2, \dots, x_n)$.

Let us select a point $\mathbf{a} = (a_1, a_2, \dots, a_n) \in X$ and a variable x_i. Now we go to the point $\mathbf{a}$ and fix all variables except x_i. That is, we introduce a function of one variable

$$\phi(x) = f(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n).$$

If the function ϕ is differentiable at a_i, then the derivative $\phi'(a_i)$ is called the **partial derivative** of f at the point $\mathbf{a}$ with respect to the variable x_i.

Notation: $\dfrac{\partial f}{\partial x_i}(\mathbf{a}),\ \dfrac{\partial}{\partial x_i} f(\mathbf{a}),\ (D_{x_i} f)(\mathbf{a}).$

Directional derivative

Consider a function $f : X \to V$ that is defined on a subset $X \subset W$ of a vector space W and takes values in a normed vector space V. For every point $\mathbf{a} \in X$ and vector $\mathbf{v} \in W$ we introduce a function of real variable $\phi(t) = f(\mathbf{a} + t\mathbf{v})$. If the function ϕ is differentiable at 0, then the derivative $\phi'(0)$ is called the **directional derivative** of f at the point $\mathbf{a}$ along the vector $\mathbf{v}$. Notation: $(D_{\mathbf{v}}f)(\mathbf{a})$.

The partial derivative is a particular case of the directional derivative, when $W = \mathbb{R}^n$ and $\mathbf{v}$ is from the standard basis.

Homogeneity $(D_{r\mathbf{v}}f)(\mathbf{a}) = r(D_{\mathbf{v}}f)(\mathbf{a})$ for all $r \in \mathbb{R}$ whenever $(D_{\mathbf{v}}f)(\mathbf{a})$ exists.

Linearity Suppose W is a normed vector space, $(D_{\mathbf{v}}f)(\mathbf{a})$ exists for all $\mathbf{v}$ and depends continuously(?) on $\mathbf{a}$. Then $\mathbf{v} \mapsto (D_{\mathbf{v}}f)(\mathbf{a})$ is a linear transformation.

Limit of a function and continuity

Let V and W be normed vector spaces. Suppose $f : E \to V$ is a function defined on a set $E \subset W$.

Definition. We say that the function f **converges to a limit** $L \in V$ at a point $\mathbf{w}_0 \in W$ if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for all $\mathbf{w} \in E$,

$$0 < \|\mathbf{w} - \mathbf{w}_0\| < \delta \quad \text{implies} \quad \|f(\mathbf{w}) - L\| < \varepsilon.$$

An equivalent condition is that for any sequence $\mathbf{w}_1, \mathbf{w}_2, \dots$ of vectors from E, $\lim_{n\to\infty} \mathbf{w}_n = \mathbf{w}_0$ implies $\lim_{n\to\infty} f(\mathbf{w}_n) = L$.

Definition. Given a set $E \subset W$, a function $f : E \to V$, and a point $\mathbf{w}_0 \in E$, the function f is **continuous at $\mathbf{w}_0$** if

$$f(\mathbf{w}_0) = \lim_{\mathbf{w}\to\mathbf{w}_0} f(\mathbf{w}).$$

We say that the function f is **continuous on** a set $E_0 \subset E$ if f is continuous at every point of E_0.

Continuity of a linear transformation

Theorem Suppose V and W are normed vector spaces and $L : W \to V$ is a linear transformation. Then the following conditions are equivalent:

(i) L is continuous everywhere on W,

(ii) L is continuous at the zero vector,

(iii) $\|L(\mathbf{w})\| \leq C\|\mathbf{w}\|$ for some $C > 0$ and all $\mathbf{w} \in W$.

Example. • If $\dim W < \infty$ then any linear transformation $L : W \to V$ is continuous. Otherwise it is not so.

The (Frechét) differential

Suppose V and W are normed vector spaces and consider a function $F : X \to V$, where $X \subset W$.

Definition. We say that the function F is **differentiable** at a point $\mathbf{a} \in X$ if it is defined in a neighborhood of $\mathbf{a}$ and there exists a continuous linear transformation $L : W \to V$ such that

$$F(\mathbf{a}+\mathbf{v}) = F(\mathbf{a}) + L(\mathbf{v}) + R(\mathbf{v}),$$

where $\|R(\mathbf{v})\|/\|\mathbf{v}\| \to 0$ as $\|\mathbf{v}\| \to 0$. The transformation L is called the **differential** of F at $\mathbf{a}$ and denoted $(DF)(\mathbf{a})$.

Theorem If a function F is differentiable at a point $\mathbf{a}$, then the directional derivatives $(D_{\mathbf{v}}F)(\mathbf{a})$ exist for all $\mathbf{v}$ and $(D_{\mathbf{v}}F)(\mathbf{a}) = (DF)(\mathbf{a})[\mathbf{v}]$.

Chain Rule If a function F is differentiable at a point $\mathbf{a}$ and a function G is differentiable at the point $\mathbf{b} = F(\mathbf{a})$, then $(D(G \circ F))(\mathbf{a}) = DG(\mathbf{b}) \circ DF(\mathbf{a})$.

Examples

• Any linear transformation $L : \mathbb{R} \to \mathbb{R}$ is a scaling $L(x) = rx$ by a scalar r. If L is the differential of a function $f : X \to \mathbb{R}$ at a point $a \in \mathbb{R}$, then $r = f'(a)$.

• Any linear transformation $L : \mathbb{R}^n \to \mathbb{R}$ is the dot product with a fixed vector, $L(\mathbf{x}) = \mathbf{x} \cdot \mathbf{v}_0$. If L is the differential of a function $f : X \to \mathbb{R}$ at a point $\mathbf{a} \in \mathbb{R}^n$, then $\mathbf{v}_0 = \nabla f(\mathbf{a})$.

• Any linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation: $L(\mathbf{x}) = B\mathbf{x}$, where $B = (b_{ij})$ is an $m \times n$ matrix. If L is the differential of a function $\mathbf{F} : X \to \mathbb{R}^m$ at a point $\mathbf{a} \in \mathbb{R}^n$, then $b_{ij} = \dfrac{\partial F_i}{\partial x_j}(\mathbf{a})$.

The matrix B of partial derivatives is called the **Jacobian matrix** of **F** and denoted $\dfrac{\partial(F_1, \dots, F_m)}{\partial(x_1, \dots, x_n)}$.

Problem. Find the directional derivative of the function $f(x,y) = e^{x+y}\sin(x-y)$ at the point $(0,0)$ along the vector $\mathbf{v} = (2,1)$.

By definition, the directional derivative $D_{\mathbf{v}}f(0,0)$ equals the derivative of the function $\phi(t) = f(t\mathbf{v})$ at the point $t = 0$. We have $\phi(t) = f(2t,t) = e^{3t}\sin t$. Then $\phi'(t) = 3e^{3t}\sin t + e^{3t}\cos t = e^{3t}(3\sin t + \cos t)$. Finally, $\phi'(0) = 1$.

Alternatively, we can find the directional derivative by the formula $D_{\mathbf{v}}f(0,0) = \nabla f(0,0)\cdot\mathbf{v}$. We obtain

$$\partial f/\partial x = e^{x+y}\sin(x-y) + e^{x+y}\cos(x-y),$$
$$\partial f/\partial y = e^{x+y}\sin(x-y) - e^{x+y}\cos(x-y).$$

Then $\nabla f(0,0) = (1,-1)$. Consequently, $D_{\mathbf{v}}f(0,0) = (1,-1)\cdot(2,1) = 1$.