

MATH 311

Topics in Applied Mathematics I

Lecture 22:

Gradient, divergence and curl.

Review of integral calculus.

Area and volume.

Gradient, divergence, and curl

Gradient of a scalar field $f = f(x_1, x_2, \dots, x_n)$ is

$$\text{grad } f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right).$$

Divergence of a vector field $\mathbf{F} = (F_1, F_2, \dots, F_n)$ is

$$\text{div } \mathbf{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \dots + \frac{\partial F_n}{\partial x_n}.$$

Curl of a vector field $\mathbf{F} = (F_1, F_2, F_3)$ is

$$\text{curl } \mathbf{F} = \left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}, \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}, \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right).$$

Informally, $\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ F_1 & F_2 & F_3 \end{vmatrix}.$

Del notation

Gradient, divergence, and curl can be denoted in a compact way using the del (a.k.a. nabla a.k.a. atled) “operator”

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right).$$

Namely, $\text{grad } f = \nabla f$, $\text{div } \mathbf{F} = \nabla \cdot \mathbf{F}$, $\text{curl } \mathbf{F} = \nabla \times \mathbf{F}$.

Theorem 1 $\text{div}(\text{curl } \mathbf{F}) = 0$ wherever the vector field \mathbf{F} is twice continuously differentiable.

Theorem 2 $\text{curl}(\text{grad } f) = \mathbf{0}$ wherever the scalar field f is twice continuously differentiable.

In the del notation, $\nabla \cdot (\nabla \times \mathbf{F}) = 0$ and $\nabla \times (\nabla f) = \mathbf{0}$.
Note that $\nabla \cdot \nabla f = \Delta f$ (the Laplacian, also denoted $\nabla^2 f$).

Problem. Find $\text{curl}(\mathbf{F})$, where

$$\mathbf{F}(x, y, z) = (x^2 + y^2)\mathbf{e}_1 + ze^{x+y}\mathbf{e}_2 + (x + \sin y)\mathbf{e}_3.$$

For any vector field $\mathbf{F} = (F_1, F_2, F_3)$ we have, informally,

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

or, formally,

$$\text{curl } \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right).$$

Problem. Find $\text{curl}(\mathbf{F})$, where

$$\mathbf{F}(x, y, z) = (x^2 + y^2)\mathbf{e}_1 + ze^{x+y}\mathbf{e}_2 + (x + \sin y)\mathbf{e}_3.$$

Let $\mathbf{G} = \text{curl } \mathbf{F}$, $\mathbf{G} = (G_1, G_2, G_3)$. We obtain

$$G_1 = \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} = \frac{\partial}{\partial y}(x + \sin y) - \frac{\partial}{\partial z}(ze^{x+y}) = \cos y - e^{x+y},$$

$$G_2 = \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} = \frac{\partial}{\partial z}(x^2 + y^2) - \frac{\partial}{\partial x}(x + \sin y) = -1,$$

$$G_3 = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{\partial}{\partial x}(ze^{x+y}) - \frac{\partial}{\partial y}(x^2 + y^2) = ze^{x+y} - 2y.$$

Hence $\mathbf{G} = \text{curl } \mathbf{F} = (\cos y - e^{x+y}, -1, ze^{x+y} - 2y)$.

Riemann sums and Riemann integral

Definition. A **Riemann sum** of a function $f : [a, b] \rightarrow \mathbb{R}$ with respect to a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$ generated by samples $t_j \in [x_{j-1}, x_j]$ is a sum

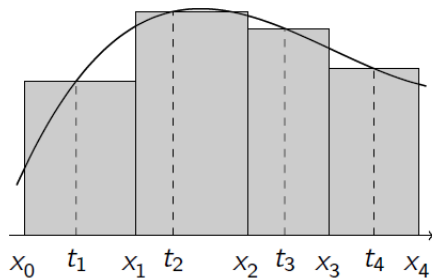
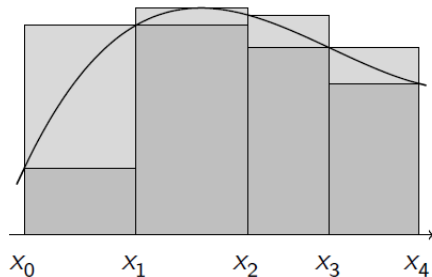
$$\mathcal{S}(f, P, t_j) = \sum_{j=1}^n f(t_j) (x_j - x_{j-1}).$$

Remark. $P = \{x_0, x_1, \dots, x_n\}$ is a partition of $[a, b]$ if $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$. The norm of the partition P is $\|P\| = \max_{1 \leq j \leq n} |x_j - x_{j-1}|$.

Definition. The Riemann sums $\mathcal{S}(f, P, t_j)$ **converge** to a limit $I(f)$ as the norm $\|P\| \rightarrow 0$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|P\| < \delta$ implies $|\mathcal{S}(f, P, t_j) - I(f)| < \varepsilon$ for any partition P and choice of samples t_j .

If this is the case, then the function f is called **integrable** on $[a, b]$ and the limit $I(f)$ is called the **integral** of f over $[a, b]$, denoted $\int_a^b f(x) dx$.

Riemann sums and Darboux sums



Integration as a linear operation

Theorem 1 If functions f, g are integrable on an interval $[a, b]$, then the sum $f + g$ is also integrable on $[a, b]$ and

$$\int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx.$$

Theorem 2 If a function f is integrable on $[a, b]$, then for each $\alpha \in \mathbb{R}$ the scalar multiple αf is also integrable on $[a, b]$ and

$$\int_a^b \alpha f(x) dx = \alpha \int_a^b f(x) dx.$$

More properties of integrals

Theorem If a function f is integrable on $[a, b]$ and $f([a, b]) \subset [A, B]$, then for each continuous function $g : [A, B] \rightarrow \mathbb{R}$ the composition $g \circ f$ is also integrable on $[a, b]$.

Theorem If functions f and g are integrable on $[a, b]$, then so is fg .

Idea of the proof. $4fg = (f + g)^2 - (f - g)^2$.

Theorem If a function f is integrable on $[a, b]$, then it is integrable on each subinterval $[c, d] \subset [a, b]$. Moreover, for any $c \in (a, b)$ we have

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

Comparison theorems for integrals

Theorem 1 If functions f, g are integrable on $[a, b]$ and $f(x) \leq g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x) dx \leq \int_a^b g(x) dx.$$

Theorem 2 If f is integrable on $[a, b]$ and $f(x) \geq 0$ for $x \in [a, b]$, then $\int_a^b f(x) dx \geq 0$.

Theorem 3 If f is integrable on $[a, b]$, then the function $|f|$ is also integrable on $[a, b]$ and

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx.$$

Fundamental theorem of calculus

Theorem If a function f is continuous on an interval $[a, b]$, then the function

$$F(x) = \int_a^x f(t) dt, \quad x \in [a, b],$$

is continuously differentiable on $[a, b]$. Moreover, $F'(x) = f(x)$ for all $x \in [a, b]$.

Theorem If a function F is differentiable on $[a, b]$ and the derivative F' is integrable on $[a, b]$, then

$$\int_a^b F'(x) dx = F(b) - F(a).$$

Problem. Evaluate $\int_0^1 \frac{x(x-3)}{(x-1)^2(x+2)} dx$.

To evaluate the integral, we need to decompose the rational function $R(x) = \frac{x(x-3)}{(x-1)^2(x+2)}$ into a sum of partial fractions:

$$\begin{aligned} R(x) &= \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{x+2} \\ &= \frac{a(x-1)(x+2) + b(x+2) + c(x-1)^2}{(x-1)^2(x+2)} \\ &= \frac{(a+c)x^2 + (a+b-2c)x + (-2a+2b+c)}{(x-1)^2(x+2)}. \end{aligned}$$

$$\begin{cases} a + c = 1 \\ a + b - 2c = -3 \\ -2a + 2b + c = 0 \end{cases}$$

Change of the variable in an integral

Theorem If ϕ is continuously differentiable on a closed interval $[a, b]$ and f is continuous on $\phi([a, b])$, then

$$\int_{\phi(a)}^{\phi(b)} f(t) dt = \int_a^b f(\phi(x)) \phi'(x) dx = \int_a^b f(\phi(x)) d\phi(x).$$

Remarks. • The **Leibniz differential** $d\phi$ of the function ϕ is defined by $d\phi(x) = \phi'(x) dx = \frac{d\phi}{dx} dx$.

• It is possible that $\phi(a) \geq \phi(b)$. Hence we set

$$\int_c^d f(t) dt = - \int_d^c f(t) dt$$

if $c > d$. Also, we set the integral to be 0 if $c = d$.

• $t = \phi(x)$ is a proper change of the variable only if the function ϕ is strictly monotone. However the theorem holds even without this assumption.

Problem. Evaluate $\int_0^{1/2} \frac{x}{\sqrt{1-x^2}} dx$.

To integrate this function, we introduce a new variable $u = 1 - x^2$:

$$\begin{aligned} \int_0^{1/2} \frac{x}{\sqrt{1-x^2}} dx &= -\frac{1}{2} \int_0^{1/2} \frac{(1-x^2)'}{\sqrt{1-x^2}} dx \\ &= -\frac{1}{2} \int_0^{1/2} \frac{1}{\sqrt{1-x^2}} d(1-x^2) = -\frac{1}{2} \int_1^{3/4} \frac{1}{\sqrt{u}} du \\ &= \int_{3/4}^1 \frac{1}{2\sqrt{u}} du = \sqrt{u} \Big|_{u=3/4}^1 = 1 - \frac{\sqrt{3}}{2}. \end{aligned}$$

Sets of measure zero

Definition. A subset E of the real line \mathbb{R} is said to have **measure zero** if for any $\varepsilon > 0$ the set E can be covered by a sequence of open intervals J_1, J_2, \dots such that $\sum_{n=1}^{\infty} |J_n| < \varepsilon$.

Examples. • Any set E that can be represented as a sequence x_1, x_2, \dots (such sets are called **countable**) has measure zero. Indeed, for any $\varepsilon > 0$, let

$$J_n = \left(x_n - \frac{\varepsilon}{2^{n+1}}, x_n + \frac{\varepsilon}{2^{n+1}} \right), \quad n = 1, 2, \dots$$

Then $E \subset J_1 \cup J_2 \cup \dots$ and $|J_n| = \varepsilon/2^n$ for all $n \in \mathbb{N}$ so that $\sum_{n=1}^{\infty} |J_n| = \varepsilon$.

- The set \mathbb{Q} of rational numbers has measure zero (since it is countable).
- Nondegenerate interval $[a, b]$ is not a set of measure zero.

Lebesgue's criterion for Riemann integrability

Definition. Suppose $P(x)$ is a property depending on $x \in S$, where $S \subset \mathbb{R}$. We say that $P(x)$ holds for **almost all** $x \in S$ (or **almost everywhere** on S) if the set $\{x \in S \mid P(x) \text{ does not hold}\}$ has measure zero.

Theorem A function $f : [a, b] \rightarrow \mathbb{R}$ is Riemann integrable on the interval $[a, b]$ if and only if f is bounded on $[a, b]$ and continuous almost everywhere on $[a, b]$.

Let \mathcal{P} be the smallest collection of subsets of \mathbb{R}^2 such that it contains all polygons and if $X, Y \in \mathcal{P}$, then $X \cup Y, X \cap Y, X \setminus Y \in \mathcal{P}$.

Theorem There exists a unique function $\mu : \mathcal{P} \rightarrow \mathbb{R}$ (called the **area function**) that satisfies the following conditions:

- **(positivity)** $\mu(X) \geq 0$ for all $X \in \mathcal{P}$;
- **(additivity)** $\mu(X \cup Y) = \mu(X) + \mu(Y)$ if $X \cap Y = \emptyset$;
- **(translation invariance)** $\mu(X + \mathbf{v}) = \mu(X)$ for all $X \in \mathcal{P}$ and $\mathbf{v} \in \mathbb{R}^2$;
- $\mu(Q) = 1$, where $Q = [0, 1] \times [0, 1]$ is the unit square.

The area function satisfies an extra condition:

- **(monotonicity)** $\mu(X) \leq \mu(Y)$ whenever $X \subset Y$.

Now for any bounded set $X \subset \mathbb{R}^2$ we let $\bar{\mu}(X) = \inf_{X \subset Y} \mu(Y)$ and $\underline{\mu}(X) = \sup_{Z \subset X} \mu(Z)$. Note that $\underline{\mu}(X) \leq \bar{\mu}(X)$. In the case of equality, the set X is called **Jordan measurable** and we let $\text{area}(X) = \bar{\mu}(X)$.

Area, volume, and determinants

- 2×2 determinants and plane geometry

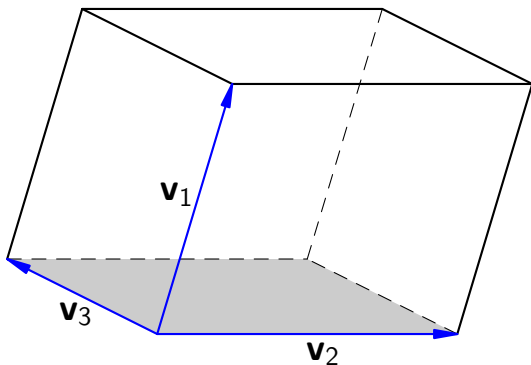
Let P be a parallelogram in the plane \mathbb{R}^2 . Suppose that vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$ are represented by adjacent sides of P . Then $\text{area}(P) = |\det A|$, where $A = (\mathbf{v}_1, \mathbf{v}_2)$, a matrix whose columns are \mathbf{v}_1 and \mathbf{v}_2 .

Consider a linear operator $L_A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by $L_A(\mathbf{v}) = A\mathbf{v}$ for any column vector \mathbf{v} . Then $\text{area}(L_A(D)) = |\det A| \text{area}(D)$ for any bounded domain D .

- 3×3 determinants and space geometry

Let Π be a parallelepiped in space \mathbb{R}^3 . Suppose that vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$ are represented by adjacent edges of Π . Then $\text{volume}(\Pi) = |\det B|$, where $B = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, a matrix whose columns are $\mathbf{v}_1, \mathbf{v}_2$, and \mathbf{v}_3 .

Similarly, $\text{volume}(L_B(D)) = |\det B| \text{volume}(D)$ for any bounded domain $D \subset \mathbb{R}^3$.



$\text{volume}(\Pi) = |\det B|$, where $B = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$. Note that the parallelepiped Π is the image under L_B of a unit cube whose adjacent edges are $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

The triple $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ obeys the right-hand rule. We say that L_B **preserves orientation** if it preserves the hand rule for any basis. This is the case if and only if $\det B > 0$.