MATH 311

Topics in Applied Mathematics I

Lecture 26:
Review for the final exam.

Topics for the final exam: Part I

Elementary linear algebra (L/C 1.1–1.5, 2.1–2.2)

- Systems of linear equations: elementary operations, Gaussian elimination, back substitution.

- Matrix of coefficients and augmented matrix. Elementary row operations, row echelon form and reduced row echelon form.

- Matrix algebra. Inverse matrix.

- Determinants: explicit formulas for 2×2 and 3×3 matrices, row and column expansions, elementary row and column operations.

Topics for the final exam: Part II

Abstract linear algebra (L/C 3.1–3.6, 4.1–4.3)

- Vector spaces (vectors, matrices, polynomials, functional spaces).
- Subspaces. Nullspace, column space, and row space of a matrix.
- Span, spanning set. Linear independence.
- Bases and dimension.
- Rank and nullity of a matrix.
- Coordinates relative to a basis.
- Change of basis, transition matrix.

- Linear transformations.
- Matrix transformations.
- Matrix of a linear mapping.
- Change of basis for a linear operator.
- Similarity of matrices.

Topics for the final exam: Part III

Advanced linear algebra (L/C 5.1–5.6, 6.1, 6.3)

- Eigenvalues, eigenvectors, eigenspaces
- Characteristic polynomial
- Bases of eigenvectors, diagonalization

- Euclidean structure in $\mathbb{R}^n$ (length, angle, dot product)
- Inner products and norms
- Orthogonal complement, orthogonal projection
- Least squares problems
- The Gram-Schmidt orthogonalization process

Topics for the final exam: Part IV

Vector analysis (L/C 8.1–8.4, 9.1–9.5, 10.1–10.3, 11.1–11.3)

- Gradient, divergence, and curl
- Fubini's Theorem
- Change of coordinates in a multiple integral
- Geometric meaning of the determinant
- Length of a curve
- Line integrals
- Green's Theorem
- Conservative vector fields
- Area of a surface
- Surface integrals
- Gauss' Theorem
- Stokes' Theorem

Problem. Consider a vector field
$\mathbf{F}(x,y,z) = (yz + 2\cos 2x,\ xz - e^z,\ xy - ye^z)$.
(i) Verify that the field $\mathbf{F}$ is conservative.

Since $\mathbf{F}$ is a smooth vector field on the entire space, it is conservative if and only if its Jacobian matrix is symmetric everywhere in $\mathbb{R}^3$. For vector fields on $\mathbb{R}^3$, this is equivalent to $\operatorname{curl}(\mathbf{F}) = \mathbf{0}$. We have to verify three identities.

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}: \quad \frac{\partial}{\partial y}(yz+2\cos 2x) = \frac{\partial}{\partial x}(xz-e^z) \iff z = z,$$

$$\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}: \quad \frac{\partial}{\partial z}(yz+2\cos 2x) = \frac{\partial}{\partial x}(xy-ye^z) \iff y = y,$$

$$\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}: \quad \frac{\partial}{\partial z}(xz - e^z) = \frac{\partial}{\partial y}(xy - ye^z)$$

$$\iff x - e^z = x - e^z.$$

Problem. Consider a vector field
$\mathbf{F}(x, y, z) = (yz + 2\cos 2x,\ xz - e^z,\ xy - ye^z)$.
(ii) Find a function f such that $\mathbf{F} = \nabla f$.

We are looking for a function $f : \mathbb{R}^3 \to \mathbb{R}$ such that
$$\frac{\partial f}{\partial x} = yz + 2\cos 2x, \quad \frac{\partial f}{\partial y} = xz - e^z, \quad \frac{\partial f}{\partial z} = xy - ye^z.$$
Integrating the first equality by x, we get
$$f(x, y, z) = \int (yz + 2\cos 2x)\, dx = xyz + \sin 2x + g(y, z).$$
Substituting this into the second equality, we obtain $xz + g'_y = xz - e^z$ so that $g'_y = -e^z$. Integrating by y, we get
$$g(y, z) = \int -e^z\, dy = -ye^z + h(z).$$
Then $f(x, y, z) = xyz + \sin 2x - ye^z + h(z)$. Substituting this into the third equality, we obtain $xy - ye^z + h'(z) = xy - ye^z$. Hence $h'(z) = 0$ so that $h(z) = c$, a constant. Finally, $f(x, y, z) = xyz + \sin 2x - ye^z + c$.

Problem. Consider a vector field
$\mathbf{F}(x, y, z) = (yz + 2\cos 2x,\ xz - e^z,\ xy - ye^z)$.
(ii) Find a function f such that $\mathbf{F} = \nabla f$.

Alternative solution: If $\mathbf{F} = \nabla f$, then

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = f(A_1) - f(A_0)$$

for any points $A_0, A_1 \in \mathbb{R}^3$ and any path $\mathbf{x}$ joining A_0 to A_1. We can use this relation to recover the function f.

For any given point $A = (x, y, z)$ we consider a linear path $\mathbf{x}_A$ from the origin to A, $\mathbf{x}_A : [0, 1] \to \mathbb{R}^3$, $\mathbf{x}_A(t) = (tx, ty, tz)$. Then

$$f(A) - f(\mathbf{0}) = \int_{\mathbf{x}_A} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 \mathbf{F}(\mathbf{x}_A(t)) \cdot \mathbf{x}_A'(t)\, dt.$$

$$f(A) - f(\mathbf{0}) = \int_{\mathbf{x}_A} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 \mathbf{F}(\mathbf{x}_A(t)) \cdot \mathbf{x}_A'(t)\, dt$$

$$= \int_0^1 (t^2yz + 2\cos 2tx,\ t^2xz - e^{tz},\ t^2xy - tye^{tz}) \cdot (x, y, z)\, dt$$

$$= \int_0^1 \big((t^2yz + 2\cos 2tx)x + (t^2xz - e^{tz})y + (t^2xy - tye^{tz})z\big)\, dt$$

$$= \int_0^1 (3t^2xyz + 2x\cos 2tx - ye^{tz} - tyze^{tz})\, dt$$

$$= t^3xyz \,\Big|_{t=0}^{1} + \sin 2tx \,\Big|_{t=0}^{1} - yte^{tz} \,\Big|_{t=0}^{1} = xyz + \sin 2x - ye^z.$$

Thus $f(x, y, z) = xyz + \sin 2x - ye^z + c$, where $c = f(\mathbf{0})$ is a constant.

Problem. Let C be a solid cylinder bounded by planes $z=0,\ z=2$ and a cylindrical surface $x^2+y^2=1$. Orient the boundary ∂C with outward normals and evaluate a surface integral

$$\oiint_{\partial C} (x^2\mathbf{e}_1 + y^2\mathbf{e}_2 + z^2\mathbf{e}_3)\cdot d\mathbf{S}.$$

By Gauss' Theorem,

$$\oiint_{\partial C} (x^2\mathbf{e}_1 + y^2\mathbf{e}_2 + z^2\mathbf{e}_3)\cdot d\mathbf{S} = \iiint_C \nabla\cdot(x^2\mathbf{e}_1 + y^2\mathbf{e}_2 + z^2\mathbf{e}_3)\,dV$$

$$= \iiint_C \left(\frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2)\right) dx\,dy\,dz$$

$$= \iiint_C 2(x+y+z)\,dx\,dy\,dz.$$

To evaluate the integral, we switch to cylindrical coordinates (r, ϕ, z) using the substitution $x = r\cos\phi$, $y = r\sin\phi$, $z = z$.

Jacobian matrix $J = \dfrac{\partial(x,y,z)}{\partial(r,\phi,z)} = \begin{pmatrix} \cos\phi & -r\sin\phi & 0 \\ \sin\phi & r\cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

$$\begin{aligned}
&\iiint_C 2(x+y+z)\,dx\,dy\,dz \\
&\quad = \int_0^2\int_0^{2\pi}\int_0^1 2(r\cos\phi + r\sin\phi + z)\big|\det J\big|\,dr\,d\phi\,dz \\
&\quad = \int_0^2\int_0^{2\pi}\int_0^1 2(r\cos\phi + r\sin\phi + z)\,r\,dr\,d\phi\,dz \\
&\quad = \int_0^2\int_0^{2\pi}\int_0^1 \Big(2r^2(\cos\phi + \sin\phi) + 2rz\Big)\,dr\,d\phi\,dz \\
&= \int_0^2\int_0^{2\pi}\int_0^1 2rz\,dr\,d\phi\,dz = 2\int_0^2 z\,dz \cdot \int_0^{2\pi} d\phi \cdot \int_0^1 r\,dr = 4\pi.
\end{aligned}$$

Problem. Let D be a region in $\mathbb{R}^3$ bounded by a paraboloid $z = x^2 + y^2$ and a plane $z = 9$. Let S denote the part of the paraboloid that bounds D, oriented by outward normals. Evaluate a surface integral

$$\iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S},$$

where $\mathbf{F}(x, y, z) = (e^{x^2+z^2}, xy + xz + yz, e^{xyz})$.

We have $\operatorname{curl} \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$

$= (xze^{xyz} - x - y, 2ze^{x^2+z^2} - yze^{xyz}, y + z)$.

Direct evaluation of the surface integral seems problematic. By Stokes' Theorem, the surface integral equals the integral of the field $\mathbf{F}$ along the circle ∂S. However evaluation of this line integral seems problematic as well.

By the corollary of Stokes' Theorem,

$$\iint_{\partial D} \operatorname{curl}(\mathbf{F})\cdot d\mathbf{S} = 0.$$

It follows that

$$\iint_{S} \operatorname{curl}(\mathbf{F})\cdot d\mathbf{S} = -\iint_{\partial D\setminus S} \operatorname{curl}(\mathbf{F})\cdot d\mathbf{S}.$$

We observe that $\partial D \setminus S$ is a horizontal disc $Q\times\{9\}$, where $Q = \{(x,y)\in\mathbb{R}^2 : x^2+y^2\leq 9\}$. It is oriented by the upward normal vector $\mathbf{n} = (0,0,1)$. Now

$$\iint_{\partial D\setminus S} \operatorname{curl}(\mathbf{F})\cdot d\mathbf{S} = \iint_{\partial D\setminus S} \operatorname{curl}(\mathbf{F})\cdot \mathbf{n}\, dS = \iint_{Q} (y+9)\, dx\, dy$$

$$= \iint_{Q} y\, dx\, dy + \iint_{Q} 9\, dx\, dy = \iint_{Q} 9\, dx\, dy = 9\operatorname{area}(Q) = 81\pi.$$

Thus $\displaystyle\iint_{S} \operatorname{curl}(\mathbf{F})\cdot d\mathbf{S} = -81\pi.$

Problem. Consider a linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v}$, where $\mathbf{v}_0 = (3/5, 0, -4/5)$.

(a) Find the matrix A of the operator L.
(b) Find the range and kernel of L.
(c) Find the eigenvalues of L.
(d) Find the matrix of the operator L^{2022} (L applied 2022 times).

$$L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v}, \quad \mathbf{v}_0 = (3/5, 0, -4/5).$$

Let $\mathbf{v} = (x, y, z) = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$. Then

$$L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 3/5 & 0 & -4/5 \\ x & y & z \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -4/5 \\ y & z \end{vmatrix} \mathbf{e}_1 - \begin{vmatrix} 3/5 & -4/5 \\ x & z \end{vmatrix} \mathbf{e}_2 + \begin{vmatrix} 3/5 & 0 \\ x & y \end{vmatrix} \mathbf{e}_3$$

$$= \tfrac{4}{5}y\mathbf{e}_1 - \left(\tfrac{4}{5}x + \tfrac{3}{5}z\right)\mathbf{e}_2 + \tfrac{3}{5}y\mathbf{e}_3 = \left(\tfrac{4}{5}y, -\tfrac{4}{5}x - \tfrac{3}{5}z, \tfrac{3}{5}y\right).$$

In particular, $L(\mathbf{e}_1) = \left(0, -\frac{4}{5}, 0\right)$, $L(\mathbf{e}_2) = \left(\frac{4}{5}, 0, \frac{3}{5}\right)$, $L(\mathbf{e}_3) = \left(0, -\frac{3}{5}, 0\right)$.

Therefore $A = \begin{pmatrix} 0 & 4/5 & 0 \\ -4/5 & 0 & -3/5 \\ 0 & 3/5 & 0 \end{pmatrix}$.

The range of the operator L is spanned by columns of the matrix A. It follows that $\mathrm{Range}(L)$ is the plane spanned by $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = (4, 0, 3)$.

The kernel of L is the nullspace of the matrix A, i.e., the solution set for the equation $A\mathbf{x} = \mathbf{0}$.

$$\begin{pmatrix} 0 & 4/5 & 0 \\ -4/5 & 0 & -3/5 \\ 0 & 3/5 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3/4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\implies x + \frac{3}{4}z = y = 0 \implies \mathbf{x} = t(-3/4, 0, 1)$.

Alternatively, the kernel of L is the set of vectors $\mathbf{v} \in \mathbb{R}^3$ such that $L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v} = \mathbf{0}$.
It follows that this is the line spanned by $\mathbf{v}_0 = (3/5, 0, -4/5)$.

Characteristic polynomial of the matrix A:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 4/5 & 0 \\ -4/5 & -\lambda & -3/5 \\ 0 & 3/5 & -\lambda \end{vmatrix}$$

$$= -\lambda^3 - (3/5)^2\lambda - (4/5)^2\lambda = -\lambda^3 - \lambda = -\lambda(\lambda^2 + 1).$$

The eigenvalues are 0, i, and $-i$.

The matrix of the operator L^{2022} is A^{2022}.

Since the matrix A has eigenvalues 0, i, and $-i$, it is diagonalizable in $\mathbb{C}^3$. Namely, $A = UDU^{-1}$, where U is an invertible matrix with complex entries and

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}.$$

Then $A^{2022} = UD^{2022}U^{-1}$. We have that $D^{2022} =$ $= \mathrm{diag}\big(0, i^{2022}, (-i)^{2022}\big) = \mathrm{diag}(0, -1, -1) = D^2$.

Hence

$$A^{2022} = UD^2U^{-1} = A^2 = \begin{pmatrix} -0.64 & 0 & -0.48 \\ 0 & -1 & 0 \\ -0.48 & 0 & -0.36 \end{pmatrix}.$$

Problem. Find the distance from the point $\mathbf{y} = (0, 0, 0, 1)$ to the subspace $V \subset \mathbb{R}^4$ spanned by vectors $\mathbf{x}_1 = (1, -1, 1, -1)$, $\mathbf{x}_2 = (1, 1, 3, -1)$, and $\mathbf{x}_3 = (-3, 7, 1, 3)$.

First we apply the Gram-Schmidt process to vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ and obtain an orthogonal basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ for the subspace V. Next we compute the orthogonal projection $\mathbf{p}$ of the vector $\mathbf{y}$ onto V:

$$\mathbf{p} = \frac{\langle \mathbf{y}, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 + \frac{\langle \mathbf{y}, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2 + \frac{\langle \mathbf{y}, \mathbf{v}_3 \rangle}{\langle \mathbf{v}_3, \mathbf{v}_3 \rangle}\mathbf{v}_3.$$

Then the distance from $\mathbf{y}$ to V equals $\|\mathbf{y} - \mathbf{p}\|$.

Alternatively, we can apply the Gram-Schmidt process to vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{y}$. We should obtain an orthogonal system $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$. Then the desired distance will be $\|\mathbf{v}_4\|$.

$$\mathbf{x}_1 = (1, -1, 1, -1),\ \mathbf{x}_2 = (1, 1, 3, -1),$$
$$\mathbf{x}_3 = (-3, 7, 1, 3),\ \mathbf{y} = (0, 0, 0, 1).$$

$$\mathbf{v}_1 = \mathbf{x}_1 = (1, -1, 1, -1),$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 = (1, 1, 3, -1) - \frac{4}{4}(1, -1, 1, -1)$$

$$= (0, 2, 2, 0),$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 - \frac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2$$

$$= (-3, 7, 1, 3) - \frac{-12}{4}(1, -1, 1, -1) - \frac{16}{8}(0, 2, 2, 0)$$

$$= (0, 0, 0, 0).$$

The Gram-Schmidt process can be used to check linear independence of vectors! It failed because the vector $\mathbf{x}_3$ is a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$. V is a plane, not a 3-dimensional subspace. To fix things, it is enough to drop $\mathbf{x}_3$, i.e., we should orthogonalize vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}$.

$$\tilde{\mathbf{v}}_3 = \mathbf{y} - \frac{\langle \mathbf{y}, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 - \frac{\langle \mathbf{y}, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2$$

$$= (0,0,0,1) - \frac{-1}{4}(1,-1,1,-1) - \frac{0}{8}(0,2,2,0)$$

$$= (1/4, -1/4, 1/4, 3/4).$$

$$\|\tilde{\mathbf{v}}_3\| = \left| \left(\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{3}{4}\right) \right| = \frac{1}{4}\,|(1,-1,1,3)| = \frac{\sqrt{12}}{4} = \frac{\sqrt{3}}{2}.$$