MATH 323

Linear Algebra

Lecture 2:
Gaussian elimination (continued).
Row echelon form.
Gauss-Jordan reduction.

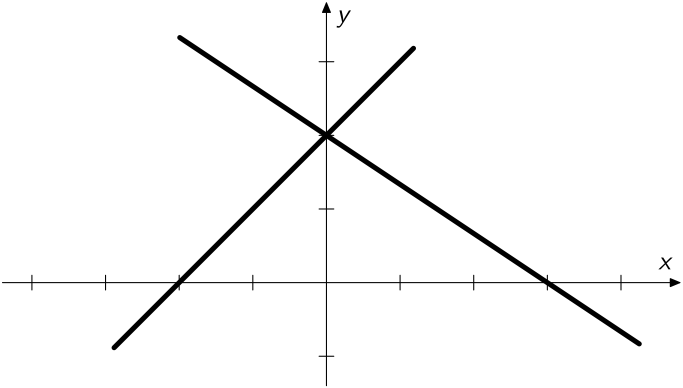
System of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Here $x_1, x_2, \ldots, x_n$ are variables and a_{ij}, b_j are constants.

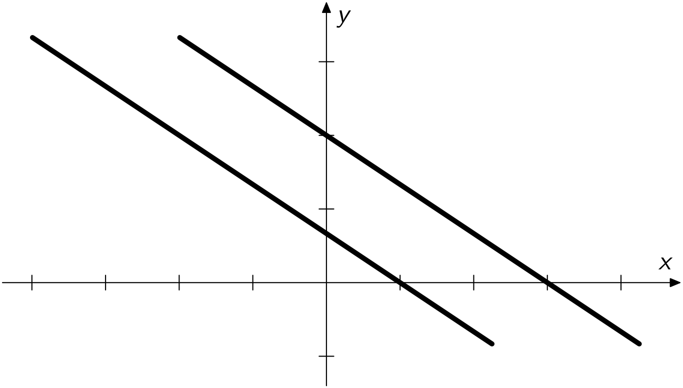
A *solution* of the system is a common solution of all equations in the system.

A system of linear equations can have **one** solution, **infinitely many** solutions, or **no** solution at all.



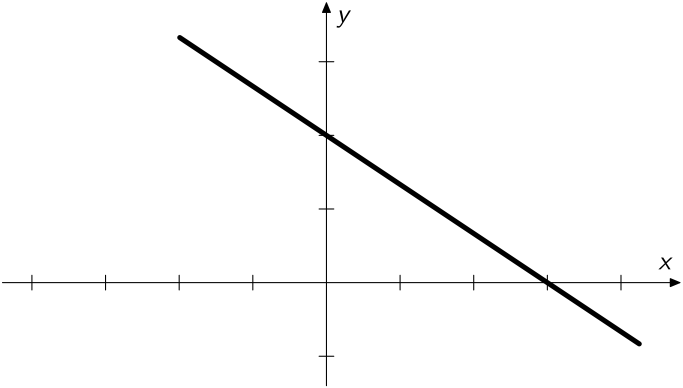
$$\begin{cases} x - y = -2 \\ 2x + 3y = 6 \end{cases}$$

$$x = 0,\ y = 2$$



$$\begin{cases} 2x + 3y = 2 \\ 2x + 3y = 6 \end{cases}$$

inconsistent system
(no solutions)



$$\begin{cases} 4x + 6y = 12 \\ 2x + 3y = 6 \end{cases} \iff 2x + 3y = 6$$

Solving systems of linear equations

Elimination method always works for systems of linear equations.

Algorithm: (1) pick a variable, solve one of the equations for it, and eliminate it from the other equations; (2) put aside the equation used in the elimination, and return to step (1).

$$x - y = 2 \implies x = y + 2$$

$$2x - y - z = 5 \implies 2(y + 2) - y - z = 5$$

After the elimination is completed, the system is solved by *back substitution*.

$$y = 1 \implies x = y + 2 = 3$$

Gaussian elimination

Gaussian elimination is a modification of the elimination method that allows only so-called *elementary operations*.

Elementary operations for systems of linear equations:

(1) to multiply an equation by a nonzero scalar;
(2) to add an equation multiplied by a scalar to another equation;
(3) to interchange two equations.

Theorem (i) Applying elementary operations to a system of linear equations does not change the solution set of the system. **(ii)** Any elementary operation can be undone by another elementary operation.

Operation 1: multiply the ith equation by $r \neq 0$.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \cdots\cdots\cdots\cdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \\ \cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

$$\Longrightarrow \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \cdots\cdots\cdots\cdots \\ (ra_{i1})x_1 + (ra_{i2})x_2 + \cdots + (ra_{in})x_n = rb_i \\ \cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

To undo the operation, multiply the ith equation by r^{-1}.

Operation 2: add r times the ith equation to the jth equation.

$$\left\{\begin{array}{c} \cdots\cdots\cdots\cdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \\ \cdots\cdots\cdots\cdots \\ a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j \\ \cdots\cdots\cdots\cdots \end{array}\right. \Longrightarrow$$

$$\left\{\begin{array}{c} \cdots\cdots\cdots\cdots \\ a_{i1}x_1 + \cdots + a_{in}x_n = b_i \\ \cdots\cdots\cdots\cdots \\ (a_{j1} + ra_{i1})x_1 + \cdots + (a_{jn} + ra_{in})x_n = b_j + rb_i \\ \cdots\cdots\cdots\cdots \end{array}\right.$$

To undo the operation, add $-r$ times the ith equation to the jth equation.

Operation 3: interchange the ith and jth equations.

$$\begin{cases} \cdots\cdots\cdots\cdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \\ \cdots\cdots\cdots\cdots \\ a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j \\ \cdots\cdots\cdots\cdots \end{cases}$$

$$\Longrightarrow \begin{cases} \cdots\cdots\cdots\cdots \\ a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j \\ \cdots\cdots\cdots\cdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \\ \cdots\cdots\cdots\cdots \end{cases}$$

To undo the operation, apply it once more.

Solution of a system of linear equations splits into two parts: **(A)** elimination and **(B)** back substitution. Both parts can be done by applying a finite number of **elementary operations**.

Example.

$$\left\{\begin{array}{rcrcrcr} x & - & y & & & = & 2 \\ 2x & - & y & - & z & = & 3 \\ x & + & y & + & z & = & 6 \end{array}\right. \to \left\{\begin{array}{rcrcrcr} x & - & y & & & = & 2 \\ & & y & - & z & = & -1 \\ & & 2y & + & z & = & 4 \end{array}\right.$$

$$\to \left\{\begin{array}{rcrcrcr} x & - & y & & & = & 2 \\ & & y & - & z & = & -1 \\ & & & & 3z & = & 6 \end{array}\right. \to \left\{\begin{array}{rcrcrcr} x & & & & & = & 3 \\ & & y & & & = & 1 \\ & & & & z & = & 2 \end{array}\right.$$

Another example.

$$\left\{\begin{array}{rcrcrcr} x & + & y & - & 2z & = & 1 \\ & & y & - & z & = & 3 \\ -x & + & 4y & - & 3z & = & 14 \end{array}\right.$$

Add the 1st equation to the 3rd equation:

$$\left\{\begin{array}{rcrcrcr} x & + & y & - & 2z & = & 1 \\ & & y & - & z & = & 3 \\ & & 5y & - & 5z & = & 15 \end{array}\right.$$

Add -5 times the 2nd equation to the 3rd equation:

$$\left\{\begin{array}{rcrcrcr} x & + & y & - & 2z & = & 1 \\ & & y & - & z & = & 3 \\ & & & & 0 & = & 0 \end{array}\right.$$

Add -1 times the 2nd equation to the 1st equation:

$$\begin{cases} x \quad\quad - z = -2 \\ \quad\quad y - z = 3 \\ \quad\quad\quad\quad 0 = 0 \end{cases} \iff \begin{cases} x = z - 2 \\ y = z + 3 \end{cases}$$

Here z is a *free variable* (x and y are *leading variables*).

It follows that $\begin{cases} x = t - 2 \\ y = t + 3 \\ z = t \end{cases}$ for some $t \in \mathbb{R}$.

System of linear equations:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 14 \end{cases}$$

Solution: $(x, y, z) = (t - 2, t + 3, t), \ \ t \in \mathbb{R}.$

In vector form, $(x, y, z) = (-2, 3, 0) + t(1, 1, 1).$

Yet another example.

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 1 \end{cases}$$

Add the 1st equation to the 3rd equation:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ 5y - 5z = 2 \end{cases}$$

Add -5 times the 2nd equation to the 3rd equation:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ 0 = -13 \end{cases}$$

System of linear equations:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 1 \end{cases}$$

Solution: no solution (*inconsistent system*).

Matrices

Definition. A *matrix* is a rectangular array of numbers.

Examples: $\begin{pmatrix} 2 & 7 \\ -1 & 0 \\ 3 & 3 \end{pmatrix}$, $\begin{pmatrix} 2 & 7 & 0.2 \\ 4.6 & 1 & 1 \end{pmatrix}$,

$\begin{pmatrix} 3/5 \\ 5/8 \\ 4 \end{pmatrix}$, $(\sqrt{2}, 0, -\sqrt{3}, 5)$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

dimensions $=$ **(# of rows)** $\times$ **(# of columns)**

n-by-n: **square matrix**

n-by-1: **column vector**

1-by-n: **row vector**

System of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \qquad \cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Coefficient matrix and column vector of the right-hand sides:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \qquad \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

System of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Augmented matrix:

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{array}\right)$$

Since the elementary operations preserve the standard form of linear equations, we can trace the solution process by looking on the **augmented matrix**.

Elementary operations for systems of linear equations correspond to **elementary row operations** for augmented matrices:

(1) to multiply a row by a nonzero scalar;
(2) to add the ith row multiplied by some $r \in \mathbb{R}$ to the jth row;
(3) to interchange two rows.

Remark. Rows are added and multiplied by scalars as vectors (namely, row vectors).

Elementary row operations

Augmented matrix:

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ \hline a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right) = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{pmatrix},$$

where $\mathbf{v}_i = (a_{i1}\ a_{i2}\ \dots\ a_{in} \,|\, b_i)$ is a row vector.

Elementary row operations

Operation 1: to multiply the ith row by $r \neq 0$:

$$\begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix} \to \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ r\mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix}$$

Elementary row operations

Operation 2: to add the ith row multiplied by r to the jth row:

$$\begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_j \\ \vdots \\ \mathbf{v}_m \end{pmatrix} \to \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_j + r\mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix}$$

Elementary row operations

Operation 3: to interchange the ith row with the jth row:

$$\begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_j \\ \vdots \\ \mathbf{v}_m \end{pmatrix} \to \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_j \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix}$$

Row echelon form

Definition. **Leading entry** of a matrix is the first nonzero entry in a row.

The goal of the Gaussian elimination is to convert the augmented matrix into **row echelon form**:

- leading entries shift to the right as we go from the first row to the last one;
- each leading entry is equal to 1.

$$\left(\begin{array}{cccccccccc|c}
1 & -1 & 3 & 0 & 2 & 1 & 4 & 0 & -3 & 7 & 2 \\
0 & 1 & 1 & -2 & 0 & 0 & 6 & 1 & 3 & 4 & 0 \\
0 & 0 & 0 & 0 & 1 & 2 & 3 & 1 & -4 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 9 & -1 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

Row echelon form

General augmented matrix in row echelon form:

$$\left(\begin{array}{cccccccccc|c}
\boxed{} & * & * & * & * & * & * & * & * & * & * \\
 & \boxed{} & * & * & * & * & * & * & * & * & * \\
 & & & & \boxed{} & * & * & * & * & * & * \\
 & & & & & & \boxed{} & * & * & * & * \\
 & & & & & & & \boxed{} & * & * & * \\
 & & & & & & & & & & \\
 & & & & & & & & & &
\end{array}\right)$$

- leading entries are boxed (all equal to 1);
- all the entries below the staircase line are zero;
- each step of the staircase has height 1;
- each circle marks a column without a leading entry that corresponds to a free variable.

Strict triangular form is a particular case of row echelon form that can occur for systems of n equations in n variables:

$$\left(\begin{array}{ccccccc|c}
\square & * & * & * & * & * & * & * \\
 & \square & * & * & * & * & * & * \\
 & & \square & * & * & * & * & * \\
 & & & \square & * & * & * & * \\
 & & & & \square & * & * & * \\
 & & & & & \square & * & * \\
 & & & & & & \square & *
\end{array}\right)$$

- no zero rows;
- no free variables.

Consistency check

The original system of linear equations is **consistent** if there is no leading entry in the rightmost column of the augmented matrix in row echelon form.

$$
\left(\begin{array}{cccccccccc|c}
\square & * & * & * & * & * & * & * & * & * & * \\
 & \square & * & * & * & * & * & * & * & * & * \\
 & & & \square & * & * & * & * & * & * & * \\
 & & & & & & \square & * & * & * & * \\
 & & & & & & & \square & * & * & * \\
 & & & & & & & & & & \square
\end{array}\right)
$$

Augmented matrix of an inconsistent system

The goal of the **Gauss-Jordan reduction** is to convert the augmented matrix into **reduced row echelon form**:

$$\left(\begin{array}{cccccccccc|c}
\boxed{1} & & * & * & & * & & & * & * & * \\
& \boxed{1} & * & * & & * & & & * & * & * \\
& & & & \boxed{1} & * & & & * & * & * \\
& & & & & & \boxed{1} & & * & * & * \\
& & & & & & & \boxed{1} & * & * & * \\
& & & & & & & & & & \\
& & & & & & & & & &
\end{array}\right)$$

- all entries below the staircase line are zero;
- each boxed entry is 1, the other entries in its column are zero;
- each circle corresponds to a free variable.

Example.

$$\begin{cases} x - y = 2 \\ 2x - y - z = 3 \\ x + y + z = 6 \end{cases} \qquad \left(\begin{array}{rrr|r} 1 & -1 & 0 & 2 \\ 2 & -1 & -1 & 3 \\ 1 & 1 & 1 & 6 \end{array}\right)$$

Row echelon form (also strict triangular):

$$\begin{cases} x - y = 2 \\ y - z = -1 \\ z = 2 \end{cases} \qquad \left(\begin{array}{rrr|r} \boxed{1} & -1 & 0 & 2 \\ 0 & \boxed{1} & -1 & -1 \\ 0 & 0 & \boxed{1} & 2 \end{array}\right)$$

Reduced row echelon form:

$$\begin{cases} x = 3 \\ y = 1 \\ z = 2 \end{cases} \qquad \left(\begin{array}{rrr|r} \boxed{1} & 0 & 0 & 3 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & \boxed{1} & 2 \end{array}\right)$$

Another example.

$$\left\{\begin{array}{rcrcrcr} x & + & y & - & 2z & = & 1 \\ & & y & - & z & = & 3 \\ -x & + & 4y & - & 3z & = & 14 \end{array}\right. \qquad \left(\begin{array}{rrr|r} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 3 \\ -1 & 4 & -3 & 14 \end{array}\right)$$

Row echelon form:

$$\left\{\begin{array}{rcrcrcr} x & + & y & - & 2z & = & 1 \\ & & y & - & z & = & 3 \\ & & & & 0 & = & 0 \end{array}\right. \qquad \left(\begin{array}{rrr|r} \boxed{1} & 1 & -2 & 1 \\ 0 & \boxed{1} & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Reduced row echelon form:

$$\left\{\begin{array}{rcrcrcr} x & & & - & z & = & -2 \\ & & y & - & z & = & 3 \\ & & & & 0 & = & 0 \end{array}\right. \qquad \left(\begin{array}{rrr|r} \boxed{1} & 0 & -1 & -2 \\ 0 & \boxed{1} & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Yet another example.

$$\begin{cases} \;\;x + \;\;y - 2z = 1 \\ \qquad\;\; y - \;\; z = 3 \\ -x + 4y - 3z = 1 \end{cases} \qquad \left(\begin{array}{rrr|r} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 3 \\ -1 & 4 & -3 & 1 \end{array}\right)$$

Row echelon form:

$$\begin{cases} x + y - 2z = 1 \\ \qquad y - \;\; z = 3 \\ \qquad\qquad\;\; 0 = 1 \end{cases} \qquad \left(\begin{array}{rrr|r} \boxed{1} & 1 & -2 & 1 \\ 0 & \boxed{1} & -1 & 3 \\ 0 & 0 & 0 & \boxed{1} \end{array}\right)$$

Reduced row echelon form:

$$\begin{cases} x \qquad - z = 0 \\ \qquad y - z = 0 \\ \qquad\quad\;\; 0 = 1 \end{cases} \qquad \left(\begin{array}{rrr|r} \boxed{1} & 0 & -1 & 0 \\ 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{array}\right)$$

How to solve a system of linear equations

- Order the variables.
- Write down the augmented matrix of the system.
- Convert the matrix to **row echelon form**.
- Check for consistency.
- Convert the matrix to **reduced row echelon form**.
- Write down the system corresponding to the reduced row echelon form.
- Determine leading and free variables.
- Rewrite the system so that the leading variables are on the left while everything else is on the right.
- Assign parameters to the free variables and write down the general solution in parametric form.

New example. $\begin{cases} x_2 + 2x_3 + 3x_4 = 6 \\ x_1 + 2x_2 + 3x_3 + 4x_4 = 10 \end{cases}$

Variables: x_1, x_2, x_3, x_4.

Augmented matrix: $\left(\begin{array}{cccc|c} 0 & 1 & 2 & 3 & 6 \\ 1 & 2 & 3 & 4 & 10 \end{array}\right)$

To get it into row echelon form, we exchange the two rows:

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 10 \\ 0 & 1 & 2 & 3 & 6 \end{array}\right)$$

Consistency check is passed. To convert into reduced row echelon form, add -2 times the 2nd row to the 1st row:

$$\left(\begin{array}{cccc|c} \boxed{1} & 0 & -1 & -2 & -2 \\ 0 & \boxed{1} & 2 & 3 & 6 \end{array}\right)$$

The leading variables are x_1 and x_2; hence x_3 and x_4 are free variables.

Back to the system:

$$\begin{cases} x_1 - x_3 - 2x_4 = -2 \\ x_2 + 2x_3 + 3x_4 = 6 \end{cases} \iff \begin{cases} x_1 = x_3 + 2x_4 - 2 \\ x_2 = -2x_3 - 3x_4 + 6 \end{cases}$$

General solution:

$$\begin{cases} x_1 = t + 2s - 2 \\ x_2 = -2t - 3s + 6 \\ x_3 = t \\ x_4 = s \end{cases} \qquad (t, s \in \mathbb{R})$$

In vector form, $(x_1, x_2, x_3, x_4) =$

$$= (-2, 6, 0, 0) + t(1, -2, 1, 0) + s(2, -3, 0, 1).$$

Example with a parameter.

$$\begin{cases} y + 3z = 0 \\ x + y - 2z = 0 \\ x + 2y + az = 0 \end{cases} \qquad (a \in \mathbb{R})$$

The system is **homogeneous** (all right-hand sides are zeros). Therefore it is consistent ($x = y = z = 0$ is a solution).

Augmented matrix: $\left(\begin{array}{ccr|c} 0 & 1 & 3 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & 2 & a & 0 \end{array}\right)$

Since the 1st row cannot serve as a pivotal one, we interchange it with the 2nd row:

$$\left(\begin{array}{ccr|c} 0 & 1 & 3 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & 2 & a & 0 \end{array}\right) \to \left(\begin{array}{ccr|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & 2 & a & 0 \end{array}\right)$$

Now we can start the elimination.
First subtract the 1st row from the 3rd row:

$$\left(\begin{array}{ccr|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & 2 & a & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & a+2 & 0 \end{array}\right)$$

The 2nd row is our new pivotal row.
Subtract the 2nd row from the 3rd row:

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & a+2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & a-1 & 0 \end{array}\right)$$

At this point row reduction splits into two cases.

Case 1: $a \neq 1$. In this case, multiply the 3rd row by $(a-1)^{-1}$:

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & a-1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} \boxed{1} & 1 & -2 & 0 \\ 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & \boxed{1} & 0 \end{array}\right)$$

The matrix is converted into row echelon form.
We proceed towards reduced row echelon form.

Subtract 3 times the 3rd row from the 2nd row:

$$\left(\begin{array}{ccr|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccr|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Add 2 times the 3rd row to the 1st row:

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Finally, subtract the 2nd row from the 1st row:

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \end{array}\right)$$

Thus $x = y = z = 0$ is the only solution.

Case 2: $a = 1$. In this case, the matrix is already in row echelon form:

$$\left(\begin{array}{ccc|c} \boxed{1} & 1 & -2 & 0 \\ 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

To get reduced row echelon form, subtract the 2nd row from the 1st row:

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} \boxed{1} & 0 & -5 & 0 \\ 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

z is a free variable.

$$\begin{cases} x - 5z = 0 \\ y + 3z = 0 \end{cases} \iff \begin{cases} x = 5z \\ y = -3z \end{cases}$$

System of linear equations:

$$\begin{cases} y + 3z = 0 \\ x + y - 2z = 0 \\ x + 2y + az = 0 \end{cases}$$

Solution: If $a \neq 1$ then $(x, y, z) = (0, 0, 0)$; if $a = 1$ then $(x, y, z) = (5t, -3t, t)$, $t \in \mathbb{R}$.