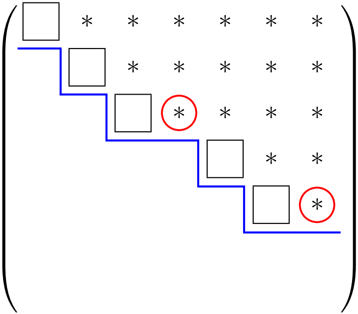
MATH 323

Linear Algebra

Lecture 3:
Row echelon form (continued).
Applications of systems of linear equations.
Matrix algebra.

Row echelon form

A matrix is in the **row echelon form** if the leading entries (equal to 1) shift to the right as we go from the first row to the last one.



- Leading entries are boxed;
- all the entries below the staircase line are zero;
- each step of the staircase has height 1;
- each circle marks a column without a leading entry.

Theorem Any matrix can be converted into row echelon form by applying elementary row operations.

Sketch of the proof: The proof is by induction on the number of columns in the matrix. It relies on the next lemma.

Lemma Any matrix can be converted to one of the following forms using elementary row operations: **(i)** $(1\ a_{12}\ a_{13}\ \dots\ a_{1n})$;

(ii) $\left(\begin{array}{c|ccc} 1 & a_{12} & \dots & a_{1n} \\ \hline 0 & & & \\ \vdots & & B & \\ 0 & & & \end{array}\right)$; **(iii)** $\left(\begin{array}{c} 1 \\ \hline 0 \\ \vdots \\ 0 \end{array}\right)$; **(iv)** $\left(\begin{array}{c|c} 0 & \\ \vdots & B \\ 0 & \end{array}\right)$; **(v)** $\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$.

In the cases (i), (iii) and (v), we already have a row echelon form. In the cases (ii) and (iv), it is enough to convert the matrix B to row echelon form. Moreover, the row reduction on the block B can be simulated by applying elementary row operations to the entire matrix.

Properties of row echelon form

Let C be a matrix in the row echelon form (resp. reduced row echelon form). We say that C is a **row echelon form** (resp. **reduced row echelon form**) of a matrix A if C can be obtained from A by applying elementary row operations.

Theorem 1 For any matrix, the reduced row echelon form exists and is unique.

Theorem 2 Suppose A and B are matrices of the same dimensions. Then the following conditions are equivalent:

(i) A and B share a reduced row echelon form;
(ii) A and B share a row echelon form;
(iii) A can be obtained from B by applying elementary row operations.

Applications of systems of linear equations

Problem 1. Find the point of intersection of the lines $x - y = -2$ and $2x + 3y = 6$ in $\mathbb{R}^2$.

$$\begin{cases} x - y = -2 \\ 2x + 3y = 6 \end{cases}$$

Problem 2. Find the point of intersection of the planes $x - y = 2$, $2x - y - z = 3$, and $x + y + z = 6$ in $\mathbb{R}^3$.

$$\begin{cases} x - y = 2 \\ 2x - y - z = 3 \\ x + y + z = 6 \end{cases}$$

Method of undetermined coefficients often involves solving systems of linear equations.

Problem 3. Find a quadratic polynomial $p(x)$ such that $p(1) = 4$, $p(2) = 3$, and $p(3) = 4$.

Suppose that $p(x) = ax^2 + bx + c$. Then $p(1) = a + b + c$, $p(2) = 4a + 2b + c$, $p(3) = 9a + 3b + c$.

$$\begin{cases} a + b + c = 4 \\ 4a + 2b + c = 3 \\ 9a + 3b + c = 4 \end{cases}$$

Method of undetermined coefficients often involves solving systems of linear equations.

Problem 3. Find a quadratic polynomial $p(x)$ such that $p(1) = 4$, $p(2) = 3$, and $p(3) = 4$.

Alternative choice of coefficients: $p(x) = \tilde{a} + \tilde{b}x + \tilde{c}x^2$.
Then $p(1) = \tilde{a} + \tilde{b} + \tilde{c}$, $p(2) = \tilde{a} + 2\tilde{b} + 4\tilde{c}$, $p(3) = \tilde{a} + 3\tilde{b} + 9\tilde{c}$.

$$\begin{cases} \tilde{a} + \tilde{b} + \tilde{c} = 4 \\ \tilde{a} + 2\tilde{b} + 4\tilde{c} = 3 \\ \tilde{a} + 3\tilde{b} + 9\tilde{c} = 4 \end{cases}$$

Problem 4. Evaluate $\displaystyle\int_{-1}^{0} \frac{x(x-3)}{(x-1)^2(x+2)}\,dx.$

To evaluate the integral, we need to decompose the rational function $R(x) = \frac{x(x-3)}{(x-1)^2(x+2)}$ into the sum of simple fractions:

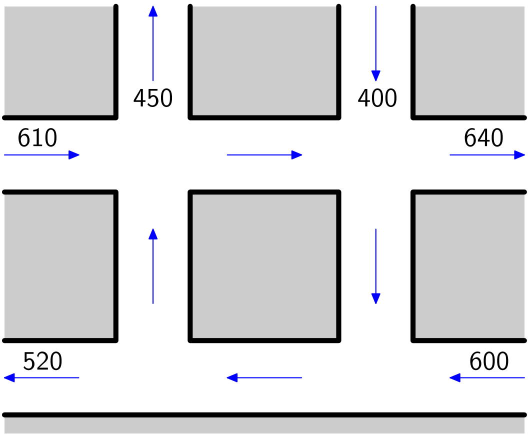
$$R(x) = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{x+2}$$

$$= \frac{a(x-1)(x+2) + b(x+2) + c(x-1)^2}{(x-1)^2(x+2)}$$

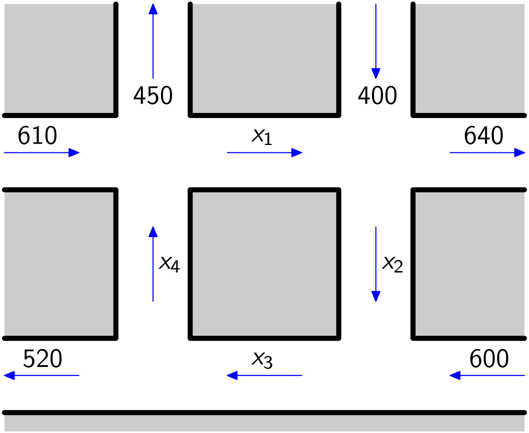
$$= \frac{(a+c)x^2 + (a+b-2c)x + (-2a+2b+c)}{(x-1)^2(x+2)}.$$

$$\begin{cases} a + c = 1 \\ a + b - 2c = -3 \\ -2a + 2b + c = 0 \end{cases}$$

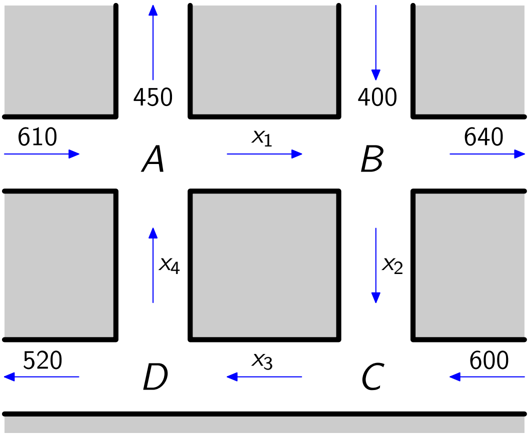
Traffic flow



Problem. Determine the amount of traffic between each of the four intersections.



$$x_1 = ?, \quad x_2 = ?, \quad x_3 = ?, \quad x_4 = ?$$



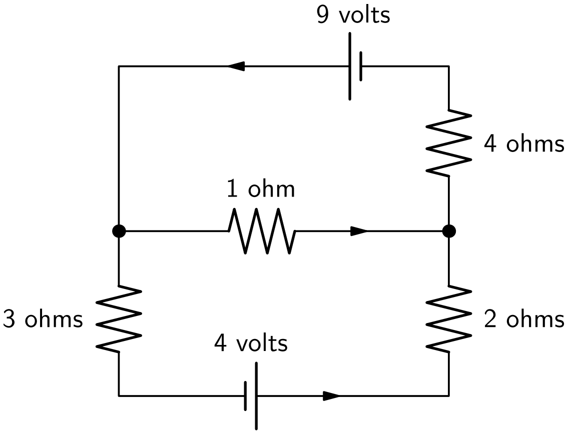
At each intersection, the incoming traffic has to match the outgoing traffic.

Intersection A: $x_4 + 610 = x_1 + 450$
Intersection B: $x_1 + 400 = x_2 + 640$
Intersection C: $x_2 + 600 = x_3$
Intersection D: $x_3 = x_4 + 520$

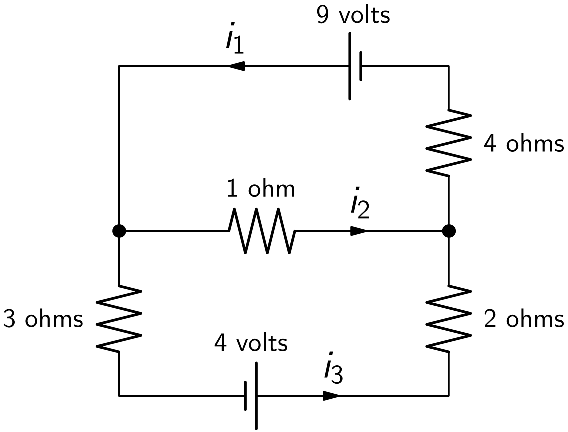
$$\begin{cases} x_4 + 610 = x_1 + 450 \\ x_1 + 400 = x_2 + 640 \\ x_2 + 600 = x_3 \\ x_3 = x_4 + 520 \end{cases}$$

$$\iff \begin{cases} -x_1 + x_4 = -160 \\ x_1 - x_2 = 240 \\ x_2 - x_3 = -600 \\ x_3 - x_4 = 520 \end{cases}$$

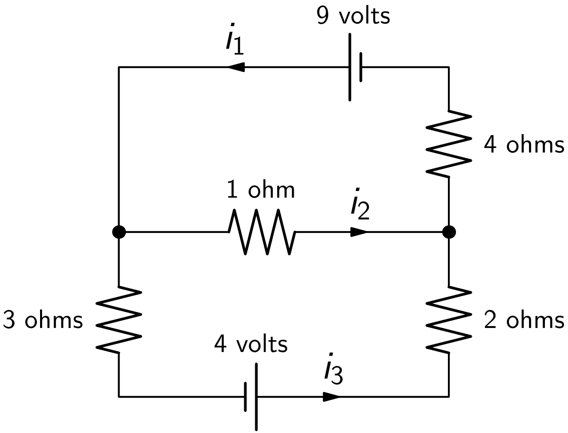
Electrical network



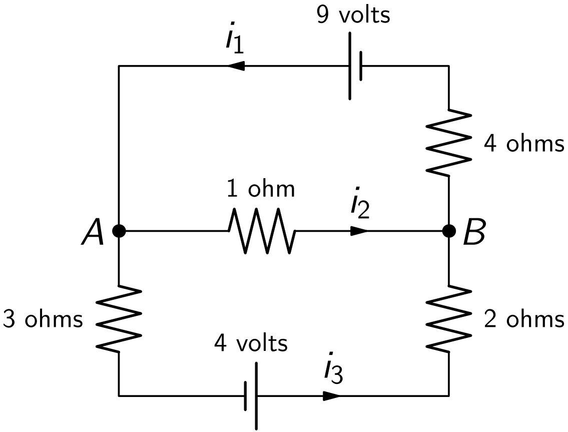
Problem. Determine the amount of current in each branch of the network.



$$i_1 = ?, \quad i_2 = ?, \quad i_3 = ?$$



Kirchhof's law #1 (junction rule): at every node the sum of the incoming currents equals the sum of the outgoing currents.



Node A: $i_1 = i_2 + i_3$

Node B: $i_2 + i_3 = i_1$

Electrical network

Kirchhof's law #2 (loop rule): around every loop the algebraic sum of all voltages is zero.

Ohm's law: for every resistor the voltage drop E, the current i, and the resistance R satisfy $E = iR$.

$$\begin{aligned} \text{Top loop:} \quad & 9 - i_2 - 4i_1 = 0 \\ \text{Bottom loop:} \quad & 4 - 2i_3 + i_2 - 3i_3 = 0 \\ \text{Big loop:} \quad & 4 - 2i_3 - 4i_1 + 9 - 3i_3 = 0 \end{aligned}$$

Remark. The 3rd equation is the sum of the first two equations.

$$\begin{cases} i_1 = i_2 + i_3 \\ 9 - i_2 - 4i_1 = 0 \\ 4 - 2i_3 + i_2 - 3i_3 = 0 \end{cases}$$

$$\iff \begin{cases} i_1 - i_2 - i_3 = 0 \\ 4i_1 + i_2 = 9 \\ -i_2 + 5i_3 = 4 \end{cases}$$

Matrices (revisited)

Definition. An **m-by-n matrix** is a rectangular array of numbers that has m rows and n columns:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

Notation: $A = (a_{ij})_{1 \le i \le n,\, 1 \le j \le m}$ or simply $A = (a_{ij})$ if the dimensions are known.

An n-dimensional vector can be represented as a $1 \times n$ matrix (row vector) or as an $n \times 1$ matrix (column vector):

$$(x_1, x_2, \ldots, x_n) \qquad \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

An $m \times n$ matrix $A = (a_{ij})$ can be regarded as a column of n-dimensional row vectors or as a row of m-dimensional column vectors:

$$A = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{pmatrix}, \qquad \mathbf{v}_i = (a_{i1}, a_{i2}, \dots, a_{in})$$

$$A = (\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n), \qquad \mathbf{w}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$$

Vector algebra

Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ be n-dimensional vectors, and $r \in \mathbb{R}$ be a scalar.

Vector sum: $\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$

Scalar multiple: $r\mathbf{a} = (ra_1, ra_2, \ldots, ra_n)$

Zero vector: $\mathbf{0} = (0, 0, \ldots, 0)$

Negative of a vector: $-\mathbf{b} = (-b_1, -b_2, \ldots, -b_n)$

Vector difference:

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) = (a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n)$$

Given n-dimensional vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ and scalars $r_1, r_2, \ldots, r_k$, the expression

$$\boxed{r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k}$$

is called a **linear combination** of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$.

Also, *vector addition* and *scalar multiplication* are called **linear operations**.

Matrix algebra

Definition. Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times n$ matrices. The **sum** $A + B$ is defined to be the $m \times n$ matrix $C = (c_{ij})$ such that $\boxed{c_{ij} = a_{ij} + b_{ij}}$ for all indices i, j.

That is, two matrices with the same dimensions can be added by adding their corresponding entries.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \\ a_{31} + b_{31} & a_{32} + b_{32} \end{pmatrix}$$

Definition. Given an $m \times n$ matrix $A = (a_{ij})$ and a number r, the **scalar multiple** rA is defined to be the $m \times n$ matrix $D = (d_{ij})$ such that $\boxed{d_{ij} = ra_{ij}}$ for all indices i, j.

That is, to multiply a matrix by a scalar r, one multiplies each entry of the matrix by r.

$$r \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} ra_{11} & ra_{12} & ra_{13} \\ ra_{21} & ra_{22} & ra_{23} \\ ra_{31} & ra_{32} & ra_{33} \end{pmatrix}$$

The $m \times n$ **zero matrix** (all entries are zeros) is denoted O_{mn} or simply O.

Negative of a matrix: $-A$ is defined as $(-1)A$.

Matrix **difference**: $A - B$ is defined as $A + (-B)$.

As far as the *linear operations* (addition and scalar multiplication) are concerned, the $m \times n$ matrices can be regarded as *mn*-dimensional vectors.

Examples

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

$$A + B = \begin{pmatrix} 5 & 2 & 0 \\ 1 & 2 & 2 \end{pmatrix}, \qquad A - B = \begin{pmatrix} 1 & 2 & -2 \\ 1 & 0 & 0 \end{pmatrix},$$

$$2C = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}, \qquad 3D = \begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix},$$

$$2C + 3D = \begin{pmatrix} 7 & 3 \\ 0 & 5 \end{pmatrix},$$

$A + D$ is not defined.