

MATH 323
Linear Algebra

Lecture 5:
Inverse matrix (continued).

Identity matrix

Definition. The **identity matrix** (or **unit matrix**) is a diagonal matrix with all diagonal entries equal to 1. The $n \times n$ identity matrix is denoted I_n or simply I .

$$I_1 = (1), \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In general,
$$I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

Theorem. Let A be an arbitrary $m \times n$ matrix. Then $I_m A = A I_n = A$.

Inverse matrix

Definition. Let A be an $n \times n$ matrix. The **inverse** of A is an $n \times n$ matrix, denoted A^{-1} , such that

$$\boxed{AA^{-1} = A^{-1}A = I.}$$

If A^{-1} exists then the matrix A is called **invertible**. Otherwise A is called **singular**.

Let A and B be $n \times n$ matrices. If A is invertible then we can **divide** B by A :

left division: $A^{-1}B$, *right division:* BA^{-1} .

Remark. There is no notation for the matrix division and the notion is not really used.

Basic properties of inverse matrices

- If $B = A^{-1}$ then $A = B^{-1}$. In other words, if A is invertible, so is A^{-1} , and $A = (A^{-1})^{-1}$.

- The inverse matrix (if it exists) is unique.

Moreover, if $AB = CA = I$ for some $n \times n$ matrices B and C , then $B = C = A^{-1}$.

Indeed, $B = IB = (CA)B = C(AB) = CI = C$.

- If $n \times n$ matrices A and B are invertible, so is AB , and $(AB)^{-1} = B^{-1}A^{-1}$.

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I,$$
$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I.$$

- By induction, $(A_1A_2 \dots A_k)^{-1} = A_k^{-1} \dots A_2^{-1}A_1^{-1}$.

Inverting diagonal matrices

Theorem A diagonal matrix $D = \text{diag}(d_1, \dots, d_n)$ is invertible if and only if all diagonal entries are nonzero: $d_i \neq 0$ for $1 \leq i \leq n$.

If D is invertible then $D^{-1} = \text{diag}(d_1^{-1}, \dots, d_n^{-1})$.

$$\begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix}^{-1} = \begin{pmatrix} d_1^{-1} & 0 & \dots & 0 \\ 0 & d_2^{-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n^{-1} \end{pmatrix}$$

Inverting diagonal matrices

Theorem A diagonal matrix $D = \text{diag}(d_1, \dots, d_n)$ is invertible if and only if all diagonal entries are nonzero: $d_i \neq 0$ for $1 \leq i \leq n$.

If D is invertible then $D^{-1} = \text{diag}(d_1^{-1}, \dots, d_n^{-1})$.

Proof: If all $d_i \neq 0$ then, clearly,

$$\text{diag}(d_1, \dots, d_n) \text{diag}(d_1^{-1}, \dots, d_n^{-1}) = \text{diag}(1, \dots, 1) = I,$$

$$\text{diag}(d_1^{-1}, \dots, d_n^{-1}) \text{diag}(d_1, \dots, d_n) = \text{diag}(1, \dots, 1) = I.$$

Now suppose that $d_i = 0$ for some i . Then for any $n \times n$ matrix B the i th row of the matrix DB is a zero row. Hence $DB \neq I$. Therefore D is singular.

Inverting 2×2 matrices

Definition. The **determinant** of a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\det A = ad - bc$.

Theorem A matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if $\det A \neq 0$.

If $\det A \neq 0$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Theorem A matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if

and only if $\det A \neq 0$. If $\det A \neq 0$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Proof: Let $B = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Then

$$AB = BA = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} = (ad - bc)I_2.$$

In the case $\det A \neq 0$, we have $A^{-1} = (\det A)^{-1}B$.

In the case $\det A = 0$, the matrix A is not invertible as

$$\text{otherwise } AB = O \implies A^{-1}(AB) = A^{-1}O = O$$

$$\implies (A^{-1}A)B = O \implies I_2B = O \implies B = O$$

$$\implies A = O, \text{ but the zero matrix is singular.}$$

Problem. Solve a system $\begin{cases} 4x + 3y = 5, \\ 3x + 2y = -1. \end{cases}$

This system is equivalent to a matrix equation $A\mathbf{x} = \mathbf{b}$,

where $A = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$.

We have $\det A = -1 \neq 0$. Hence A is invertible.

$$\begin{aligned} A\mathbf{x} = \mathbf{b} &\implies A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b} \implies (A^{-1}A)\mathbf{x} = A^{-1}\mathbf{b} \\ &\implies \mathbf{x} = A^{-1}\mathbf{b}. \end{aligned}$$

Conversely, $\mathbf{x} = A^{-1}\mathbf{b} \implies A\mathbf{x} = A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = \mathbf{b}$.

Thus the system has a unique solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ -1 \end{pmatrix} = \frac{1}{-1} \begin{pmatrix} 2 & -3 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ -1 \end{pmatrix} = \begin{pmatrix} -13 \\ 19 \end{pmatrix}.$$

System of n linear equations in n variables:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases} \iff \mathbf{Ax} = \mathbf{b},$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Theorem If the matrix A is invertible then the system has a unique solution, which is $\mathbf{x} = A^{-1}\mathbf{b}$.

General results on inverse matrices

Theorem 1 Given an $n \times n$ matrix A , the following conditions are equivalent:

- (i) A is invertible;
- (ii) $\mathbf{x} = \mathbf{0}$ is the only solution of the matrix equation $A\mathbf{x} = \mathbf{0}$;
- (iii) for some n -dimensional column vector \mathbf{b} , the matrix equation $A\mathbf{x} = \mathbf{b}$ has a unique solution (which is $\mathbf{x} = A^{-1}\mathbf{b}$);
- (iv) the matrix equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for any n -dimensional column vector \mathbf{b} ;
- (v) the row echelon form of A has no zero rows;
- (vi) the reduced row echelon form of A is the identity matrix.

Theorem 2 Suppose that a sequence of elementary row operations converts a matrix A into the identity matrix.

Then the same sequence of operations converts the identity matrix into the inverse matrix A^{-1} .

Row echelon form of a square matrix:

$$\begin{pmatrix} \square & * & * & * & * & * & * \\ & \square & * & * & * & * & * \\ & & \square & * & * & * & * \\ & & & \square & * & * & * \\ & & & & \square & * & * \\ & & & & & \square & * \\ & & & & & & \square \end{pmatrix}$$

invertible case

$$\begin{pmatrix} \square & * & * & * & * & * & * \\ & \square & * & * & * & * & * \\ & & \square & * & * & * & * \\ & & & \square & * & * & * \\ & & & & \square & * & * \\ & & & & & \square & * \\ & & & & & & \square \end{pmatrix}$$

noninvertible case

For any matrix in row echelon form, the number of columns with leading entries equals the number of rows with leading entries. For a square matrix, also the number of columns *without* leading entries (i.e., the number of free variables in a related system of linear equations) equals the number of rows *without* leading entries (i.e., zero rows).

Row echelon form of a square matrix:

A 7x7 matrix in row echelon form. The main diagonal elements are represented by empty squares. All other elements are asterisks. A blue line starts at the top-left square and moves down and right in a staircase pattern, ending at the bottom-right square. This indicates that every row has a pivot element.

invertible case

A 7x7 matrix in row echelon form. The main diagonal elements are represented by empty squares. All other elements are asterisks. A blue line starts at the top-left square and moves down and right in a staircase pattern, ending at the bottom-right square. However, the asterisks at the positions (3,4), (4,5), and (6,7) are circled in red. This indicates that these rows have zero pivots, making the matrix noninvertible.

noninvertible case

Hence the row echelon form of a square matrix A is either strict triangular or else it has a zero row. In the former case, the equation $A\mathbf{x} = \mathbf{b}$ *always* has a unique solution. In the latter case, $A\mathbf{x} = \mathbf{b}$ *never* has a unique solution. Also, in the former case the reduced row echelon form of A is I .

Example. $A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}.$

To check whether A is invertible, we convert it to row echelon form.

Interchange the 1st row with the 2nd row:

$$\begin{pmatrix} 1 & 0 & 1 \\ 3 & -2 & 0 \\ -2 & 3 & 0 \end{pmatrix}$$

Add -3 times the 1st row to the 2nd row:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & -3 \\ -2 & 3 & 0 \end{pmatrix}$$

Add 2 times the 1st row to the 3rd row:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & -3 \\ 0 & 3 & 2 \end{pmatrix}$$

Multiply the 2nd row by -0.5 :

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1.5 \\ 0 & 3 & 2 \end{pmatrix}$$

Add -3 times the 2nd row to the 3rd row:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1.5 \\ 0 & 0 & -2.5 \end{pmatrix}$$

Multiply the 3rd row by -0.4 :

$$\begin{pmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 1.5 \\ 0 & 0 & \boxed{1} \end{pmatrix}$$

We already know that the matrix A is invertible.

Let's proceed towards reduced row echelon form.

Add -1.5 times the 3rd row to the 2nd row:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Add -1 times the 3rd row to the 1st row:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

To obtain A^{-1} , we need to apply the following sequence of elementary row operations to the identity matrix:

- interchange the 1st row with the 2nd row,
- add -3 times the 1st row to the 2nd row,
- add 2 times the 1st row to the 3rd row,
- multiply the 2nd row by -0.5 ,
- add -3 times the 2nd row to the 3rd row,
- multiply the 3rd row by -0.4 ,
- add -1.5 times the 3rd row to the 2nd row,
- add -1 times the 3rd row to the 1st row.

A convenient way to compute the inverse matrix A^{-1} is to merge the matrices A and I into one 3×6 matrix $(A | I)$, and apply elementary row operations to this new matrix.

$$A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(A | I) = \left(\begin{array}{ccc|ccc} 3 & -2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array} \right)$$

$$\left(\begin{array}{ccc|ccc} 3 & -2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array} \right)$$

Interchange the 1st row with the 2nd row:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 3 & -2 & 0 & 1 & 0 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array} \right)$$

Add -3 times the 1st row to the 2nd row:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -2 & -3 & 1 & -3 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array} \right)$$

Add 2 times the 1st row to the 3rd row:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -2 & -3 & 1 & -3 & 0 \\ 0 & 3 & 2 & 0 & 2 & 1 \end{array} \right)$$

Multiply the 2nd row by -0.5 :

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1.5 & -0.5 & 1.5 & 0 \\ 0 & 3 & 2 & 0 & 2 & 1 \end{array} \right)$$

Add -3 times the 2nd row to the 3rd row:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1.5 & -0.5 & 1.5 & 0 \\ 0 & 0 & -2.5 & 1.5 & -2.5 & 1 \end{array} \right)$$

Multiply the 3rd row by -0.4 :

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1.5 & -0.5 & 1.5 & 0 \\ 0 & 0 & 1 & -0.6 & 1 & -0.4 \end{array} \right)$$

Add -1.5 times the 3rd row to the 2nd row:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0.4 & 0 & 0.6 \\ 0 & 0 & 1 & -0.6 & 1 & -0.4 \end{array} \right)$$

Add -1 times the 3rd row to the 1st row:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.6 & 0 & 0.4 \\ 0 & 1 & 0 & 0.4 & 0 & 0.6 \\ 0 & 0 & 1 & -0.6 & 1 & -0.4 \end{array} \right) = (I | A^{-1})$$

Thus
$$\begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{3}{5} & 0 & \frac{2}{5} \\ \frac{2}{5} & 0 & \frac{3}{5} \\ -\frac{3}{5} & 1 & -\frac{2}{5} \end{pmatrix}.$$

That is,

$$\begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{5} & 0 & \frac{2}{5} \\ \frac{2}{5} & 0 & \frac{3}{5} \\ -\frac{3}{5} & 1 & -\frac{2}{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} \frac{3}{5} & 0 & \frac{2}{5} \\ \frac{2}{5} & 0 & \frac{3}{5} \\ -\frac{3}{5} & 1 & -\frac{2}{5} \end{pmatrix} \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Why does it work?

Converting the matrix $(A|I) = \left(\begin{array}{ccc|ccc} 3 & -2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array} \right)$

into reduced row echelon form is equivalent to converting three matrices

$$\left(\begin{array}{ccc|c} 3 & -2 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ -2 & 3 & 0 & 0 \end{array} \right), \left(\begin{array}{ccc|c} 3 & -2 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ -2 & 3 & 0 & 0 \end{array} \right), \left(\begin{array}{ccc|c} 3 & -2 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -2 & 3 & 0 & 1 \end{array} \right).$$

The latter are augmented matrices of certain systems of linear equations. In the matrix form, $A\mathbf{x} = \mathbf{e}_1$, $A\mathbf{x} = \mathbf{e}_2$, and $A\mathbf{x} = \mathbf{e}_3$, where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are consecutive columns of I .

Suppose column vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are solutions of these systems and let $B = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$. Then

$$AB = A(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = (A\mathbf{v}_1, A\mathbf{v}_2, A\mathbf{v}_3) = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = I.$$

Row reduction and matrix algebra

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ 2b_1 & 2b_2 & 2b_3 \\ c_1 & c_2 & c_3 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1+3a_1 & b_2+3a_2 & b_3+3a_3 \\ c_1 & c_2 & c_3 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{pmatrix}.$$

Theorem Any elementary row operation can be simulated as left multiplication by a certain matrix.

Elementary matrices

$$E = \begin{pmatrix} 1 & & & & & & & \\ \vdots & \ddots & & & & & & \\ 0 & \cdots & 1 & & & & & \\ \vdots & & \vdots & \ddots & & & & \\ 0 & \cdots & r & \cdots & 1 & & & \\ \vdots & & \vdots & & \vdots & \ddots & & \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 & \end{pmatrix} \begin{array}{l} \text{row \#}i \\ \\ \text{row \#}j \end{array}$$

To obtain the matrix EA from A , add r times the i th row to the j th row. To obtain the matrix AE from A , add r times the j th column to the i th column.

Why does it work? (continued)

Assume that a square matrix A can be converted to the identity matrix by a sequence of elementary row operations. Then $E_k E_{k-1} \dots E_2 E_1 A = I$, where E_1, E_2, \dots, E_k are elementary matrices simulating those operations.

Applying the same sequence of operations to the identity matrix, we obtain the matrix

$$B = E_k E_{k-1} \dots E_2 E_1 I = E_k E_{k-1} \dots E_2 E_1.$$

Thus $BA = I$. Besides, B is invertible since elementary matrices are invertible. It follows that $A = B^{-1}$, then $B = A^{-1}$.