

MATH 323  
Linear Algebra

**Lecture 15:**  
**Linear transformations (continued).**  
**Range and kernel.**  
**General linear equations.**

## Linear transformation

*Definition.* Given vector spaces  $V_1$  and  $V_2$ , a mapping  $L : V_1 \rightarrow V_2$  is **linear** if

$$L(\mathbf{x} + \mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y}),$$

$$L(r\mathbf{x}) = rL(\mathbf{x})$$

for any  $\mathbf{x}, \mathbf{y} \in V_1$  and  $r \in \mathbb{R}$ .

## Basic properties of linear transformations

Let  $L : V_1 \rightarrow V_2$  be a linear mapping.

- $L(r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k) = r_1L(\mathbf{v}_1) + \cdots + r_kL(\mathbf{v}_k)$   
for all  $k \geq 1$ ,  $\mathbf{v}_1, \dots, \mathbf{v}_k \in V_1$ , and  $r_1, \dots, r_k \in \mathbb{R}$ .

$$L(r_1\mathbf{v}_1 + r_2\mathbf{v}_2) = L(r_1\mathbf{v}_1) + L(r_2\mathbf{v}_2) = r_1L(\mathbf{v}_1) + r_2L(\mathbf{v}_2),$$

$$L(r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + r_3\mathbf{v}_3) = L(r_1\mathbf{v}_1 + r_2\mathbf{v}_2) + L(r_3\mathbf{v}_3) = \\ = r_1L(\mathbf{v}_1) + r_2L(\mathbf{v}_2) + r_3L(\mathbf{v}_3), \text{ and so on.}$$

- $L(\mathbf{0}_1) = \mathbf{0}_2$ , where  $\mathbf{0}_1$  and  $\mathbf{0}_2$  are zero vectors in  $V_1$  and  $V_2$ , respectively.

$$L(\mathbf{0}_1) = L(0\mathbf{0}_1) = 0L(\mathbf{0}_1) = \mathbf{0}_2.$$

- $L(-\mathbf{v}) = -L(\mathbf{v})$  for any  $\mathbf{v} \in V_1$ .

$$L(-\mathbf{v}) = L((-1)\mathbf{v}) = (-1)L(\mathbf{v}) = -L(\mathbf{v}).$$

## Examples of linear mappings

- *Scaling*  $L : V \rightarrow V$ ,  $L(\mathbf{v}) = s\mathbf{v}$ , where  $s \in \mathbb{R}$ .

- *Dot product with a fixed vector*

$\ell : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $\ell(\mathbf{v}) = \mathbf{v} \cdot \mathbf{v}_0$ , where  $\mathbf{v}_0 \in \mathbb{R}^n$ .

- *Cross product with a fixed vector*

$L : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ ,  $L(\mathbf{v}) = \mathbf{v} \times \mathbf{v}_0$ , where  $\mathbf{v}_0 \in \mathbb{R}^3$ .

- *Multiplication by a fixed matrix*

$L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $L(\mathbf{v}) = A\mathbf{v}$ , where  $A$  is an  $m \times n$  matrix and all vectors are column vectors.

- *Coordinate mapping*

$L : V \rightarrow \mathbb{R}^n$ ,  $L(\mathbf{v}) =$  coordinates of  $\mathbf{v}$  relative to an ordered basis  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  for the vector space  $V$ .

## Linear mappings of functional vector spaces

- *Evaluation at a fixed point*

$$\ell : F(\mathbb{R}) \rightarrow \mathbb{R}, \quad \ell(f) = f(a), \quad \text{where } a \in \mathbb{R}.$$

- *Multiplication by a fixed function*

$$L : F(\mathbb{R}) \rightarrow F(\mathbb{R}), \quad L(f) = gf, \quad \text{where } g \in F(\mathbb{R}).$$

- *Differentiation*  $D : C^1(\mathbb{R}) \rightarrow C(\mathbb{R}), \quad L(f) = f'.$

- *Integration over a finite interval*

$$\ell : C(\mathbb{R}) \rightarrow \mathbb{R}, \quad \ell(f) = \int_a^b f(x) dx, \quad \text{where}$$

$a, b \in \mathbb{R}, \quad a < b.$

## More properties of linear mappings

- If a linear mapping  $L : V \rightarrow W$  is invertible then the inverse mapping  $L^{-1} : W \rightarrow V$  is also linear.
- If  $L : V \rightarrow W$  and  $M : W \rightarrow X$  are linear mappings then the composition  $M \circ L : V \rightarrow X$  is also linear.
- If  $L_1 : V \rightarrow W$  and  $L_2 : V \rightarrow W$  are linear mappings then the sum  $L_1 + L_2$  is also linear.

## Linear differential operators

- an ordinary differential operator

$$L : C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R}), \quad L = g_0 \frac{d^2}{dx^2} + g_1 \frac{d}{dx} + g_2,$$

where  $g_0, g_1, g_2$  are smooth functions on  $\mathbb{R}$ .

That is,  $L(f) = g_0 f'' + g_1 f' + g_2 f$ .

- Laplace's operator

$$\Delta : C^\infty(\mathbb{R}^2) \rightarrow C^\infty(\mathbb{R}^2), \quad \Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

(a.k.a. the Laplacian; also denoted by  $\nabla^2$ ).

## Linear integral operators

- anti-derivative

$$L : C[a, b] \rightarrow C^1[a, b], \quad (Lf)(x) = \int_a^x f(y) dy.$$

- Hilbert-Schmidt operator

$$L : C[a, b] \rightarrow C[c, d], \quad (Lf)(x) = \int_a^b K(x, y)f(y) dy,$$

where  $K \in C([c, d] \times [a, b])$ .

- Laplace transform

$$\mathcal{L} : BC(0, \infty) \rightarrow C(0, \infty), \quad (\mathcal{L}f)(x) = \int_0^{\infty} e^{-xy} f(y) dy.$$



*Examples.*  $\mathcal{M}_{m,n}(\mathbb{R})$ : the space of  $m \times n$  matrices.

- $\alpha : \mathcal{M}_{m,n}(\mathbb{R}) \rightarrow \mathcal{M}_{n,m}(\mathbb{R}), \alpha(A) = A^T.$

$$\alpha(A + B) = \alpha(A) + \alpha(B) \iff (A + B)^T = A^T + B^T.$$

$$\alpha(rA) = r\alpha(A) \iff (rA)^T = rA^T.$$

Hence  $\alpha$  is linear.

- $\beta : \mathcal{M}_{2,2}(\mathbb{R}) \rightarrow \mathbb{R}, \beta(A) = \det A.$

Let  $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$  and  $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$

Then  $A + B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$

We have  $\det(A) = \det(B) = 0$  while  $\det(A + B) = 1.$

Hence  $\beta(A + B) \neq \beta(A) + \beta(B)$  so that  $\beta$  is not linear.

## Range and kernel

Let  $V, W$  be vector spaces and  $L : V \rightarrow W$  be a linear mapping.

*Definition.* The **range** (or **image**) of  $L$  is the set of all vectors  $\mathbf{w} \in W$  such that  $\mathbf{w} = L(\mathbf{v})$  for some  $\mathbf{v} \in V$ . The range of  $L$  is denoted  $L(V)$ .

The **kernel** of  $L$ , denoted  $\ker L$ , is the set of all vectors  $\mathbf{v} \in V$  such that  $L(\mathbf{v}) = \mathbf{0}$ .

**Theorem** (i) The range of  $L$  is a subspace of  $W$ .  
(ii) The kernel of  $L$  is a subspace of  $V$ .

*Example.*  $L : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ ,  $L \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & -1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ .

The kernel  $\ker(L)$  is the nullspace of the matrix.

$$L \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + y \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

The range  $L(\mathbb{R}^3)$  is the column space of the matrix.

*Example.*  $L : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ ,  $L \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & -1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ .

The range of  $L$  is spanned by vectors  $(1, 1, 1)$ ,  $(0, 2, 0)$ , and  $(-1, -1, -1)$ . It follows that  $L(\mathbb{R}^3)$  is the plane spanned by  $(1, 1, 1)$  and  $(0, 1, 0)$ .

To find  $\ker(L)$ , we apply row reduction to the matrix:

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & -1 \\ 1 & 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence  $(x, y, z) \in \ker(L)$  if  $x - z = y = 0$ .

It follows that  $\ker(L)$  is the line spanned by  $(1, 0, 1)$ .

## More examples

- $f : \mathcal{M}_{2,2}(\mathbb{R}) \rightarrow \mathcal{M}_{2,2}(\mathbb{R}), f(A) = A + A^T.$

$$f \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2a & b+c \\ b+c & 2d \end{pmatrix}.$$

$\ker f$  is the subspace of anti-symmetric matrices, the range of  $f$  is the subspace of symmetric matrices.

- $g : \mathcal{M}_{2,2}(\mathbb{R}) \rightarrow \mathcal{M}_{2,2}(\mathbb{R}), g(A) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} A.$

$$g \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}.$$

The range of  $g$  is the subspace of matrices with the zero second row,  $\ker g$  is the same as the range

$$\implies g(g(A)) = O.$$

$\mathcal{P}$ : the space of polynomials.

$\mathcal{P}_n$ : the space of polynomials of degree less than  $n$ .

- $D : \mathcal{P} \rightarrow \mathcal{P}, (Dp)(x) = p'(x).$

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n$$

$$\implies (Dp)(x) = a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1}$$

The range of  $D$  is the entire  $\mathcal{P}$ ,  $\ker D = \mathcal{P}_1 =$  the subspace of constants.

- $D : \mathcal{P}_4 \rightarrow \mathcal{P}_4, (Dp)(x) = p'(x).$

$$p(x) = ax^3 + bx^2 + cx + d \implies (Dp)(x) = 3ax^2 + 2bx + c$$

The range of  $D$  is  $\mathcal{P}_3$ ,  $\ker D = \mathcal{P}_1$ .

*Example.*  $L: C^3(\mathbb{R}) \rightarrow C(\mathbb{R})$ ,  $L(u) = u''' - 2u'' + u'$ .

According to the theory of differential equations, the initial value problem

$$\begin{cases} u'''(x) - 2u''(x) + u'(x) = g(x), & x \in \mathbb{R}, \\ u(a) = b_0, \\ u'(a) = b_1, \\ u''(a) = b_2 \end{cases}$$

has a unique solution for any  $g \in C(\mathbb{R})$  and any  $b_0, b_1, b_2 \in \mathbb{R}$ . It follows that  $L(C^3(\mathbb{R})) = C(\mathbb{R})$ .

Also, the initial data evaluation  $I(u) = (u(a), u'(a), u''(a))$ , which is a linear mapping  $I: C^3(\mathbb{R}) \rightarrow \mathbb{R}^3$ , is one-to-one when restricted to  $\ker(L)$ . Hence  $\dim \ker(L) = 3$ .

It is easy to check that  $L(xe^x) = L(e^x) = L(1) = 0$ .

Besides, the functions  $xe^x$ ,  $e^x$ , and 1 are linearly independent (use Wronskian). It follows that  $\ker(L) = \text{Span}(xe^x, e^x, 1)$ .

## General linear equations

*Definition.* A **linear equation** is an equation of the form

$$L(\mathbf{x}) = \mathbf{b},$$

where  $L : V \rightarrow W$  is a linear mapping,  $\mathbf{b}$  is a given vector from  $W$ , and  $\mathbf{x}$  is an unknown vector from  $V$ .

The range of  $L$  is the set of all vectors  $\mathbf{b} \in W$  such that the equation  $L(\mathbf{x}) = \mathbf{b}$  has a solution.

The kernel of  $L$  is the solution set of the **homogeneous** linear equation  $L(\mathbf{x}) = \mathbf{0}$ .

**Theorem** If the linear equation  $L(\mathbf{x}) = \mathbf{b}$  is solvable and  $\dim \ker L < \infty$ , then the general solution is

$$\mathbf{x}_0 + t_1 \mathbf{v}_1 + \cdots + t_k \mathbf{v}_k,$$

where  $\mathbf{x}_0$  is a particular solution,  $\mathbf{v}_1, \dots, \mathbf{v}_k$  is a basis for the kernel of  $L$ , and  $t_1, \dots, t_k$  are arbitrary scalars.



*Example.* 
$$\begin{cases} x + y + z = 4, \\ x + 2y = 3. \end{cases}$$

$$L: \mathbb{R}^3 \rightarrow \mathbb{R}^2, \quad L \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Linear equation:  $L(\mathbf{x}) = \mathbf{b}$ , where  $\mathbf{b} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ .

$$\left( \begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 2 & 0 & 3 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & -1 & -1 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 1 & -1 & -1 \end{array} \right)$$

$$\begin{cases} x + 2z = 5 \\ y - z = -1 \end{cases} \iff \begin{cases} x = 5 - 2z \\ y = -1 + z \end{cases}$$

$$(x, y, z) = (5 - 2t, -1 + t, t) = (5, -1, 0) + t(-2, 1, 1).$$

*Example.*  $u'''(x) - 2u''(x) + u'(x) = e^{2x}$ .

Linear operator  $L : C^3(\mathbb{R}) \rightarrow C(\mathbb{R})$ ,

$$Lu = u''' - 2u'' + u'.$$

Linear equation:  $Lu = b$ , where  $b(x) = e^{2x}$ .

We already know that functions  $xe^x$ ,  $e^x$  and 1 form a basis for the kernel of  $L$ . It remains to find a particular solution.

$$L(e^{2x}) = 8e^{2x} - 2(4e^{2x}) + 2e^{2x} = 2e^{2x}.$$

Since  $L$  is a linear operator,  $L\left(\frac{1}{2}e^{2x}\right) = e^{2x}$ .

Particular solution:  $u_0(x) = \frac{1}{2}e^{2x}$ .

Thus the general solution is

$$u(x) = \frac{1}{2}e^{2x} + t_1xe^x + t_2e^x + t_3.$$