MATH 323

Linear Algebra

Lecture 16:
Matrix transformations.
Matrix of a linear transformation.
Similarity of matrices.

Linear transformation

Definition. Given vector spaces V_1 and V_2, a mapping $L : V_1 \to V_2$ is **linear** if

$$\boxed{L(\mathbf{x}+\mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y}),}$$

$$\boxed{L(r\mathbf{x}) = rL(\mathbf{x})}$$

for any $\mathbf{x}, \mathbf{y} \in V_1$ and $r \in \mathbb{R}$.

Basic properties of linear mappings:

- $L(r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k) = r_1L(\mathbf{v}_1) + \cdots + r_kL(\mathbf{v}_k)$ for all $k \geq 1$, $\mathbf{v}_1, \ldots, \mathbf{v}_k \in V_1$, and $r_1, \ldots, r_k \in \mathbb{R}$.
- $L(\mathbf{0}_1) = \mathbf{0}_2$, where $\mathbf{0}_1$ and $\mathbf{0}_2$ are zero vectors in V_1 and V_2, respectively.
- $L(-\mathbf{v}) = -L(\mathbf{v})$ for any $\mathbf{v} \in V_1$.

Matrix transformations

Any $m \times n$ matrix A gives rise to a transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ given by $L(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^n$ and $L(\mathbf{x}) \in \mathbb{R}^m$ are regarded as column vectors. This transformation is **linear**.

Example. $$L \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 4 & 7 \\ 0 & 5 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Let $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$ be the standard basis for $\mathbb{R}^3$. We have that $L(\mathbf{e}_1) = (1, 3, 0)$, $L(\mathbf{e}_2) = (0, 4, 5)$, $L(\mathbf{e}_3) = (2, 7, 8)$. Thus $L(\mathbf{e}_1), L(\mathbf{e}_2), L(\mathbf{e}_3)$ are columns of the matrix.

Problem. Find a linear mapping $L : \mathbb{R}^3 \to \mathbb{R}^2$ such that $L(\mathbf{e}_1) = (1, 1)$, $L(\mathbf{e}_2) = (0, -2)$, $L(\mathbf{e}_3) = (3, 0)$, where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is the standard basis for $\mathbb{R}^3$.

$$L(x, y, z) = L(x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3)$$
$$= xL(\mathbf{e}_1) + yL(\mathbf{e}_2) + zL(\mathbf{e}_3)$$
$$= x(1, 1) + y(0, -2) + z(3, 0) = (x + 3z, x - 2y)$$

$$L(x, y, z) = \begin{pmatrix} x + 3z \\ x - 2y \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 \\ 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Columns of the matrix are vectors $L(\mathbf{e}_1), L(\mathbf{e}_2), L(\mathbf{e}_3)$.

Theorem Suppose $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map. Then there exists an $m \times n$ matrix A such that $L(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. Columns of A are vectors $L(\mathbf{e}_1), L(\mathbf{e}_2), \dots, L(\mathbf{e}_n)$, where $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ is the standard basis for $\mathbb{R}^n$.

$$\mathbf{y} = A\mathbf{x} \iff \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\iff \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \dots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

Let V and W be vector spaces and S be a subset of V.

Theorem (i) If S spans V, then any linear transformation $L : V \to W$ is uniquely determined by its restriction to S.

(ii) If S is linearly independent then any function $L : S \to W$ can be extended to a linear transformation from V to W.

(iii) If S is a basis for V then any function $L : S \to W$ can be uniquely extended to a linear transformation from V to W.

Idea of the proof: If $\mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_n\mathbf{v}_n$, where $\mathbf{v}_i \in S$, $r_i \in \mathbb{R}$, then $L(\mathbf{v}) = r_1 L(\mathbf{v}_1) + r_2 L(\mathbf{v}_2) + \cdots + r_n L(\mathbf{v}_n)$ for any linear map $L : V \to W$.

Change of coordinates (revisited)

Let V be a vector space.

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be a basis for V and $g_1 : V \to \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.

Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be another basis for V and $g_2 : V \to \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.

$$\begin{array}{ccccc} & & V & & \\ & {}^{g_1}\swarrow & & \searrow^{g_2} & \\ \mathbb{R}^n & & \longrightarrow & & \mathbb{R}^n \end{array}$$

The composition $g_2 \circ g_1^{-1}$ is a linear mapping of $\mathbb{R}^n$ to itself. Hence it's represented as $\mathbf{x} \mapsto U\mathbf{x}$, where U is an $n \times n$ matrix.

U is called the **transition matrix** from $\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_n$ to $\mathbf{u}_1, \mathbf{u}_2 \ldots, \mathbf{u}_n$. Columns of U are coordinates of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ with respect to the basis $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$.

Matrix of a linear transformation

Let V, W be vector spaces and $f : V \to W$ be a linear map.

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be a basis for V and $g_1 : V \to \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.

Let $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ be a basis for W and $g_2 : W \to \mathbb{R}^m$ be the coordinate mapping corresponding to this basis.

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ g_1 \big\downarrow & & \big\downarrow g_2 \\ \mathbb{R}^n & \longrightarrow & \mathbb{R}^m \end{array}$$

The composition $g_2 \circ f \circ g_1^{-1}$ is a linear mapping of $\mathbb{R}^n$ to $\mathbb{R}^m$. Hence it's represented as $\mathbf{x} \mapsto A\mathbf{x}$, where A is an $m \times n$ matrix.

A is called the **matrix of** f with respect to bases $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and $\mathbf{w}_1, \ldots, \mathbf{w}_m$. Columns of A are coordinates of vectors $f(\mathbf{v}_1), \ldots, f(\mathbf{v}_n)$ with respect to the basis $\mathbf{w}_1, \ldots, \mathbf{w}_m$.

Examples. • $D : \mathcal{P}_3 \to \mathcal{P}_2$, $(Dp)(x) = p'(x)$.

Let A_D be the matrix of D with respect to the bases $1, x, x^2$ and $1, x$. Columns of A_D are coordinates of polynomials $D1$, Dx, Dx^2 w.r.t. the basis $1, x$.

$$D1 = 0,\ Dx = 1,\ Dx^2 = 2x \implies A_D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

• $L : \mathcal{P}_3 \to \mathcal{P}_3$, $(Lp)(x) = p(x+1)$.

Let A_L be the matrix of L w.r.t. the basis $1, x, x^2$.
$L1 = 1$, $Lx = 1 + x$, $Lx^2 = (x+1)^2 = 1 + 2x + x^2$.

$$\implies A_L = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem. Consider a linear operator L on the vector space of 2×2 matrices given by

$$L\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix}.$$

Find the matrix of L with respect to the basis

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let M_L denote the desired matrix.

It follows from the definition that M_L is a 4×4 matrix whose columns are coordinates of the matrices

$$L(E_1), L(E_2), L(E_3), L(E_4)$$

with respect to the basis E_1, E_2, E_3, E_4.

$$L(E_1) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix} = 1E_1 + 0E_2 + 3E_3 + 0E_4,$$

$$L(E_2) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix} = 0E_1 + 1E_2 + 0E_3 + 3E_4,$$

$$L(E_3) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 4 & 0 \end{pmatrix} = 2E_1 + 0E_2 + 4E_3 + 0E_4,$$

$$L(E_4) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 4 \end{pmatrix} = 0E_1 + 2E_2 + 0E_3 + 4E_4.$$

Therefore

$$M_L = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{pmatrix}.$$

Thus the relation

$$\begin{pmatrix} x_1 & y_1 \\ z_1 & w_1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

is equivalent to the relation

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}.$$

Problem. Consider a linear operator $L : \mathbb{R}^2 \to \mathbb{R}^2$,

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Find the matrix of L with respect to the basis $\mathbf{v}_1 = (3, 1)$, $\mathbf{v}_2 = (2, 1)$.

Let N be the desired matrix. Columns of N are coordinates of the vectors $L(\mathbf{v}_1)$ and $L(\mathbf{v}_2)$ w.r.t. the basis $\mathbf{v}_1, \mathbf{v}_2$.

$$L(\mathbf{v}_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \quad L(\mathbf{v}_2) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

Clearly, $L(\mathbf{v}_2) = \mathbf{v}_1 = 1\mathbf{v}_1 + 0\mathbf{v}_2$.

$$L(\mathbf{v}_1) = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2 \iff \begin{cases} 3\alpha + 2\beta = 4 \\ \alpha + \beta = 1 \end{cases} \iff \begin{cases} \alpha = 2 \\ \beta = -1 \end{cases}$$

Thus $N = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$.

Change of basis for a linear operator

Let $L : V \to V$ be a linear operator on a vector space V.

Let A be the matrix of L relative to a basis $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ for V. Let B be the matrix of L relative to another basis $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n$ for V.

Let U be the transition matrix from the basis $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ to $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n$.

$$
\begin{array}{ccc}
\boxed{\mathbf{a}\text{-coordinates of } \mathbf{v}} & \xrightarrow{A} & \boxed{\mathbf{a}\text{-coordinates of } L(\mathbf{v})} \\
U \big\downarrow & & \big\downarrow U \\
\boxed{\mathbf{b}\text{-coordinates of } \mathbf{v}} & \xrightarrow{B} & \boxed{\mathbf{b}\text{-coordinates of } L(\mathbf{v})}
\end{array}
$$

It follows that $UA\mathbf{x} = BU\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n \implies UA = BU$.
Then $A = U^{-1}BU$ and $B = UAU^{-1}$.

Problem. Consider a linear operator $L : \mathbb{R}^2 \to \mathbb{R}^2$,

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Find the matrix of L with respect to the basis $\mathbf{v}_1 = (3,1)$, $\mathbf{v}_2 = (2,1)$.

Let S be the matrix of L with respect to the standard basis, N be the matrix of L with respect to the basis $\mathbf{v}_1, \mathbf{v}_2$, and U be the transition matrix from $\mathbf{v}_1, \mathbf{v}_2$ to $\mathbf{e}_1, \mathbf{e}_2$. Then $N = U^{-1}SU$.

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix},$$

$$N = U^{-1}SU = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}.$$

Similarity of matrices

Definition. An $n\times n$ matrix B is said to be **similar** to an $n\times n$ matrix A if $\boxed{B = S^{-1}AS}$ for some nonsingular $n\times n$ matrix S.

Remark. Two $n\times n$ matrices are similar if and only if they represent the same linear operator on $\mathbb{R}^n$ with respect to different bases.

Theorem Similarity is an *equivalence relation*, which means that **(i)** any square matrix A is similar to itself;
(ii) if B is similar to A, then A is similar to B;
(iii) if A is similar to B and B is similar to C, then A is similar to C.

Corollary The set of $n\times n$ matrices is partitioned into disjoint subsets (called *similarity classes*) such that all matrices in the same subset are similar to each other while matrices from different subsets are never similar.

Theorem Similarity is an *equivalence relation*, i.e.,
(i) any square matrix A is similar to itself;
(ii) if B is similar to A, then A is similar to B;
(iii) if A is similar to B and B is similar to C, then A is similar to C.

Proof: **(i)** $A = I^{-1}AI$.

(ii) If $B = S^{-1}AS$ then $A = SBS^{-1} = (S^{-1})^{-1}BS^{-1}$
$= S_1^{-1}BS_1$, where $S_1 = S^{-1}$.

(iii) If $A = S^{-1}BS$ and $B = T^{-1}CT$ then
$A = S^{-1}(T^{-1}CT)S = (S^{-1}T^{-1})C(TS) = (TS)^{-1}C(TS)$
$= S_2^{-1}CS_2$, where $S_2 = TS$.

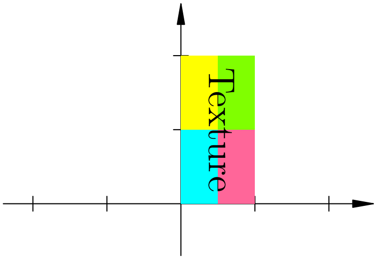
Theorem If A and B are similar matrices then they have the same **(i)** determinant, **(ii)** trace = the sum of diagonal entries, **(iii)** rank, and **(iv)** nullity.

Linear transformations of $\mathbb{R}^2$

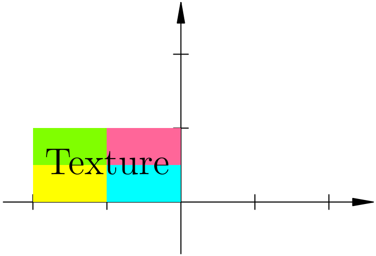
Any linear mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ is represented as multiplication of a 2-dimensional column vector by a 2×2 matrix: $f(\mathbf{x}) = A\mathbf{x}$ or

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

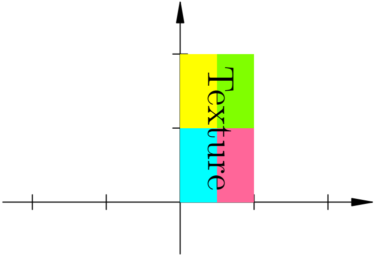
Linear transformations corresponding to particular matrices can have various geometric properties.



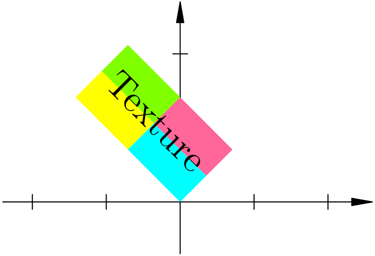
$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



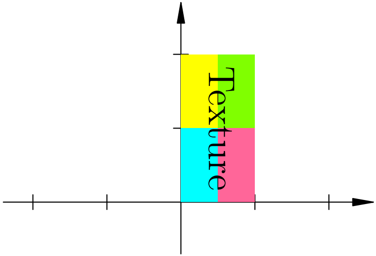
Rotation by 90^o



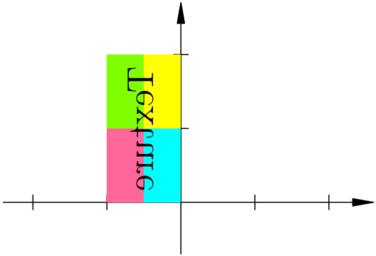
$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$



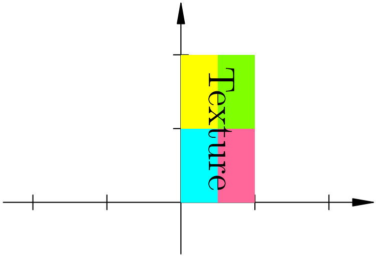
Rotation by 45°



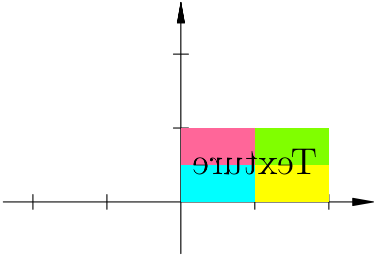
$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$



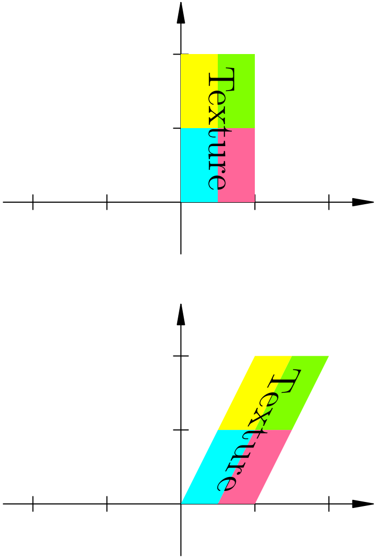
Reflection about the vertical axis



$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

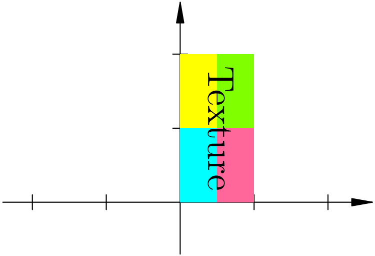


Reflection about the line $x - y = 0$

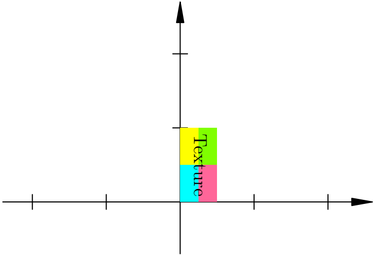


$$A = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}$$

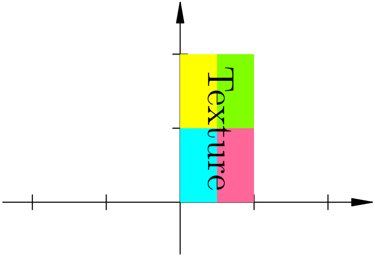
Horizontal shear



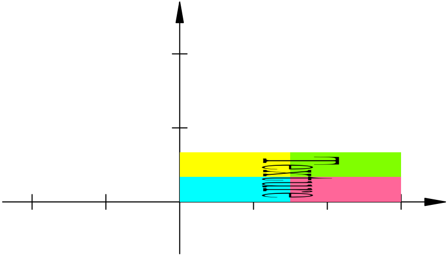
$$A = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$



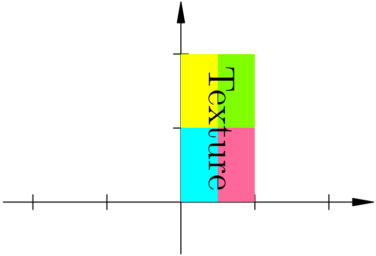
Scaling



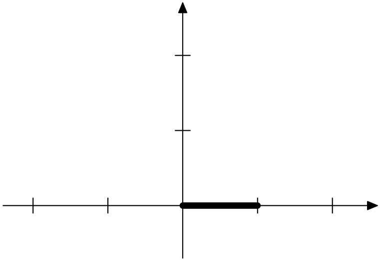
$$A = \begin{pmatrix} 3 & 0 \\ 0 & 1/3 \end{pmatrix}$$



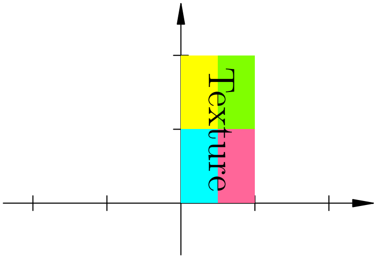
Squeeze



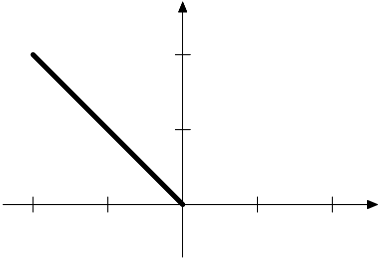
$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$



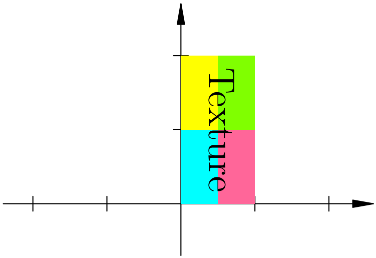
Vertical projection on the horizontal axis



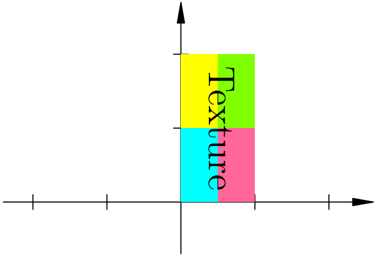
$$A = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$$



Horizontal projection
on the line $x + y = 0$



$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



Identity