MATH 323

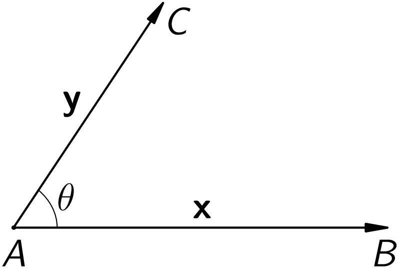
Linear Algebra

Lecture 18:
Orthogonal projection.
Least squares problems.

Euclidean structure

In addition to the linear structure (addition and scaling), space $\mathbb{R}^3$ carries the Euclidean structure:

- length of a vector: $|\mathbf{x}|$,
- angle between vectors: θ,
- dot product: $\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}|\,|\mathbf{y}|\cos\theta$.



Length and distance

Definition. The **length** of a vector $\mathbf{v} = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$ is

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

The **distance** between vectors $\mathbf{x}$ and $\mathbf{y}$ (or between points with the same coordinates) is $\|\mathbf{y} - \mathbf{x}\|$.

Properties of length:

$\|\mathbf{x}\| \geq 0$, $\|\mathbf{x}\| = 0$ only if $\mathbf{x} = \mathbf{0}$ (positivity)

$\|r\mathbf{x}\| = |r|\,\|\mathbf{x}\|$ (homogeneity)

$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (triangle inequality)

Scalar product

Definition. The **scalar product** of vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ is

$$\boxed{\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.}$$

Properties of scalar product:

$\mathbf{x} \cdot \mathbf{x} \geq 0$, $\mathbf{x} \cdot \mathbf{x} = 0$ only if $\mathbf{x} = \mathbf{0}$ (positivity)

$\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$ (symmetry)

$(\mathbf{x} + \mathbf{y}) \cdot \mathbf{z} = \mathbf{x} \cdot \mathbf{z} + \mathbf{y} \cdot \mathbf{z}$ (distributive law)

$(r\mathbf{x}) \cdot \mathbf{y} = r(\mathbf{x} \cdot \mathbf{y})$ (homogeneity)

In particular, $\mathbf{x} \cdot \mathbf{y}$ is a **bilinear** function (i.e., it is both a linear function of $\mathbf{x}$ and a linear function of $\mathbf{y}$).

Angle

Cauchy-Schwarz inequality: $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \, \|\mathbf{y}\|$.

By the Cauchy-Schwarz inequality, for any nonzero vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ we have

$$\cos\theta = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\| \, \|\mathbf{y}\|} \quad \text{for a unique} \quad 0 \leq \theta \leq \pi.$$

θ is called the **angle** between the vectors **x** and **y**.

The vectors **x** and **y** are said to be **orthogonal** (denoted $\mathbf{x} \perp \mathbf{y}$) if $\mathbf{x} \cdot \mathbf{y} = 0$ (i.e., if $\theta = 90^{\circ}$).

Orthogonality

Definition 1. Vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are said to be **orthogonal** (denoted $\mathbf{x} \perp \mathbf{y}$) if $\boxed{\mathbf{x} \cdot \mathbf{y} = 0.}$

Definition 2. A vector $\mathbf{x} \in \mathbb{R}^n$ is said to be **orthogonal** to a nonempty set $Y \subset \mathbb{R}^n$ (denoted $\mathbf{x} \perp Y$) if $\mathbf{x} \cdot \mathbf{y} = 0$ for any $\mathbf{y} \in Y$.

Definition 3. Nonempty sets $X, Y \subset \mathbb{R}^n$ are said to be **orthogonal** (denoted $X \perp Y$) if $\mathbf{x} \cdot \mathbf{y} = 0$ for any $\mathbf{x} \in X$ and $\mathbf{y} \in Y$.

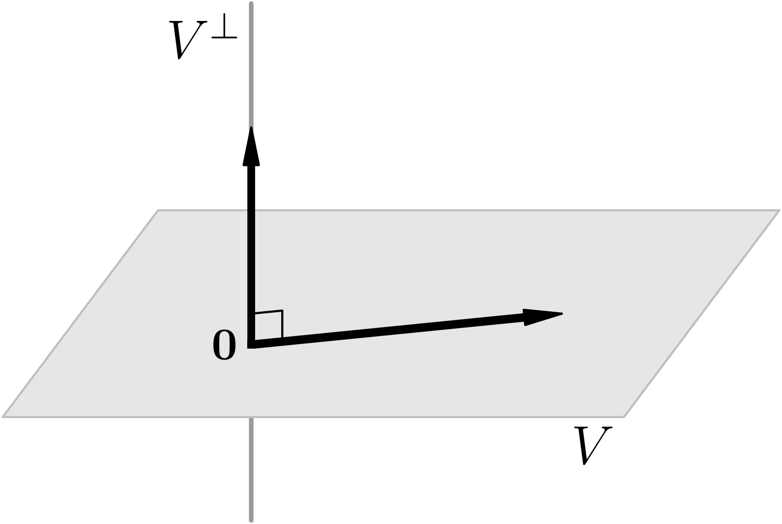
Orthogonal complement

Definition. Let $S \subset \mathbb{R}^n$ be a nonempty set. The **orthogonal complement** of S, denoted $S^\perp$, is the set of all vectors $\mathbf{x} \in \mathbb{R}^n$ that are orthogonal to S.

Theorem 1 (i) $S^\perp$ is a subspace of $\mathbb{R}^n$.
(ii) $S^\perp = \mathrm{Span}(S)^\perp$.

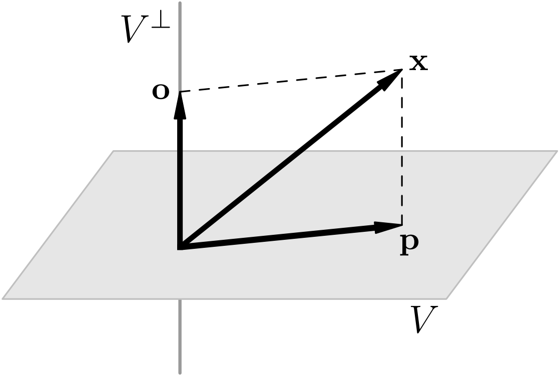
Theorem 2 If V is a subspace of $\mathbb{R}^n$, then
(i) $(V^\perp)^\perp = V$,
(ii) $V \cap V^\perp = \{\mathbf{0}\}$,
(iii) $\dim V + \dim V^\perp = n$.

Theorem 3 If V is the row space of a matrix, then $V^\perp$ is the nullspace of the same matrix.


$V^\perp$
$\mathbf{0}$
V

Orthogonal projection

Theorem Let V be a subspace of $\mathbb{R}^n$. Then any vector $\mathbf{x} \in \mathbb{R}^n$ is uniquely represented as $\mathbf{x} = \mathbf{p} + \mathbf{o}$, where $\mathbf{p} \in V$ and $\mathbf{o} \in V^\perp$.



The component $\mathbf{p}$ is called the **orthogonal projection** of the vector $\mathbf{x}$ onto the subspace V.

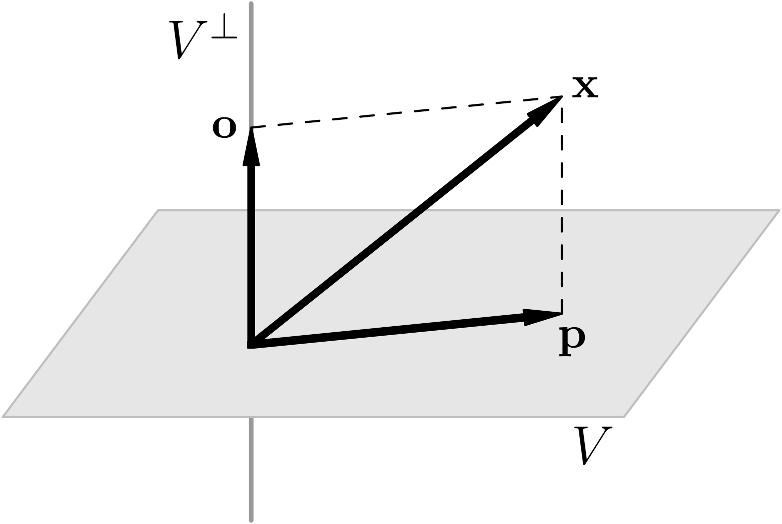
Theorem Let V be a subspace of $\mathbb{R}^n$. Then any vector $\mathbf{x} \in \mathbb{R}^n$ is uniquely represented as $\mathbf{x} = \mathbf{p} + \mathbf{o}$, where $\mathbf{p} \in V$ and $\mathbf{o} \in V^\perp$.

Proof of uniqueness: Suppose $\mathbf{x} = \mathbf{p} + \mathbf{o} = \mathbf{p}' + \mathbf{o}'$, where $\mathbf{p}, \mathbf{p}' \in V$ and $\mathbf{o}, \mathbf{o}' \in V^\perp$. Then $\mathbf{p} - \mathbf{p}' = \mathbf{o}' - \mathbf{o}$. Since $\mathbf{p} - \mathbf{p}' \in V$, $\mathbf{o}' - \mathbf{o} \in V^\perp$, and $V \cap V^\perp = \{\mathbf{0}\}$, it follows that $\mathbf{p} - \mathbf{p}' = \mathbf{o}' - \mathbf{o} = \mathbf{0}$. Thus $\mathbf{p}' = \mathbf{p}$ and $\mathbf{o}' = \mathbf{o}$.

Theorem Let V be a subspace of $\mathbb{R}^n$. Then any vector $\mathbf{x} \in \mathbb{R}^n$ is uniquely represented as $\mathbf{x} = \mathbf{p} + \mathbf{o}$, where $\mathbf{p} \in V$ and $\mathbf{o} \in V^\perp$.

Proof of existence: Let $\mathbf{v}_1, \dots, \mathbf{v}_k$ be a basis for V and $\mathbf{w}_1, \dots, \mathbf{w}_m$ be a basis for $V^\perp$. We claim that vectors $\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{w}_1, \dots, \mathbf{w}_m$ are linearly independent. Indeed, assume that $r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k + s_1\mathbf{w}_1 + \cdots + s_m\mathbf{w}_m = \mathbf{0}$ for some scalars r_i, s_j. Then $\mathbf{v} + \mathbf{w} = \mathbf{0} = \mathbf{0} + \mathbf{0}$, where $\mathbf{v} = r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k$ is in V and $\mathbf{w} = s_1\mathbf{w}_1 + \cdots + s_m\mathbf{w}_m$ is in $V^\perp$. By uniqueness (already proven!), $\mathbf{v} = \mathbf{w} = \mathbf{0}$. Consequently, $r_1 = \cdots = r_k = 0$ and $s_1 = \cdots = s_m = 0$.

Notice that $k + m = \dim V + \dim V^\perp = n$. Therefore linear independence of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{w}_1, \dots, \mathbf{w}_m$ implies that they form a basis for $\mathbb{R}^n$. Now for any vector $\mathbf{x} \in \mathbb{R}^n$ we have an expansion $\mathbf{x} = \alpha_1\mathbf{v}_1 + \cdots + \alpha_k\mathbf{v}_k + \beta_1\mathbf{w}_1 + \cdots + \beta_m\mathbf{w}_m$ $= \mathbf{p} + \mathbf{o}$, where $\mathbf{p} = \alpha_1\mathbf{v}_1 + \cdots + \alpha_k\mathbf{v}_k$ is in V and $\mathbf{o} = \beta_1\mathbf{w}_1 + \cdots + \beta_m\mathbf{w}_m$ is in $V^\perp$.


$V^\perp$
$\mathbf{x}$
$\mathbf{o}$
$\mathbf{p}$
V

Let V be a subspace of $\mathbb{R}^n$. Suppose $\mathbf{p}$ is the orthogonal projection of a vector $\mathbf{x} \in \mathbb{R}^n$ onto V.

Theorem $\|\mathbf{x}-\mathbf{v}\| > \|\mathbf{x}-\mathbf{p}\|$ for any $\mathbf{v} \neq \mathbf{p}$ in V.

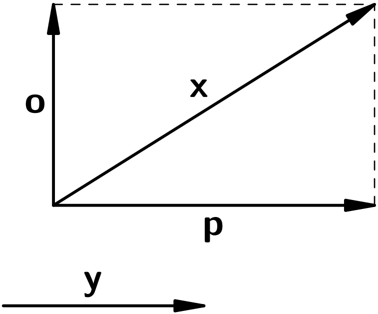
Remark. Thus $\|\mathbf{x}-\mathbf{p}\| = \min\limits_{\mathbf{v}\in V} \|\mathbf{x}-\mathbf{v}\|$ is the distance from the vector $\mathbf{x}$ to the subspace V.

Proof: Let $\mathbf{o} = \mathbf{x}-\mathbf{p}$; then $\mathbf{o} \in V^{\perp}$. Further, let $\mathbf{o}_1 = \mathbf{x}-\mathbf{v}$, and $\mathbf{v}_1 = \mathbf{p}-\mathbf{v}$. We have $\mathbf{o}_1 = \mathbf{o}+\mathbf{v}_1$, $\mathbf{v}_1 \in V$, and $\mathbf{v}_1 \neq \mathbf{0}$. Since $\mathbf{o} \perp V$, it follows that $\mathbf{o}\cdot\mathbf{v}_1 = 0$.

$$\begin{aligned} \|\mathbf{o}_1\|^2 = \mathbf{o}_1\cdot\mathbf{o}_1 &= (\mathbf{o}+\mathbf{v}_1)\cdot(\mathbf{o}+\mathbf{v}_1) \\ &= \mathbf{o}\cdot\mathbf{o}+\mathbf{v}_1\cdot\mathbf{o}+\mathbf{o}\cdot\mathbf{v}_1+\mathbf{v}_1\cdot\mathbf{v}_1 \\ &= \mathbf{o}\cdot\mathbf{o}+\mathbf{v}_1\cdot\mathbf{v}_1 = \|\mathbf{o}\|^2+\|\mathbf{v}_1\|^2 > \|\mathbf{o}\|^2. \end{aligned}$$

Orthogonal projection onto a vector

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, with $\mathbf{y} \neq \mathbf{0}$.
Then there exists a unique decomposition $\mathbf{x} = \mathbf{p} + \mathbf{o}$ such that $\mathbf{p}$ is parallel to $\mathbf{y}$ and $\mathbf{o}$ is orthogonal to $\mathbf{y}$.



$\mathbf{p}$ = orthogonal projection of $\mathbf{x}$ onto $\mathbf{y}$

Orthogonal projection onto a vector

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, with $\mathbf{y} \neq \mathbf{0}$.
Then there exists a unique decomposition $\mathbf{x} = \mathbf{p} + \mathbf{o}$ such that $\mathbf{p}$ is parallel to $\mathbf{y}$ and $\mathbf{o}$ is orthogonal to $\mathbf{y}$.

We have $\mathbf{p} = \alpha\mathbf{y}$ for some $\alpha \in \mathbb{R}$. Then

$$0 = \mathbf{o} \cdot \mathbf{y} = (\mathbf{x} - \alpha\mathbf{y}) \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{y} - \alpha\mathbf{y} \cdot \mathbf{y}.$$

$$\implies \alpha = \frac{\mathbf{x} \cdot \mathbf{y}}{\mathbf{y} \cdot \mathbf{y}} \implies \boxed{\mathbf{p} = \frac{\mathbf{x} \cdot \mathbf{y}}{\mathbf{y} \cdot \mathbf{y}}\,\mathbf{y}}$$

Problem. Find the distance from the point $\mathbf{x} = (3, 1)$ to the line spanned by $\mathbf{y} = (2, -1)$.

Consider the decomposition $\mathbf{x} = \mathbf{p} + \mathbf{o}$, where $\mathbf{p}$ is parallel to $\mathbf{y}$ while $\mathbf{o} \perp \mathbf{y}$. The required distance is the length of the orthogonal component $\mathbf{o}$.

$$\mathbf{p} = \frac{\mathbf{x} \cdot \mathbf{y}}{\mathbf{y} \cdot \mathbf{y}} \mathbf{y} = \frac{5}{5} (2, -1) = (2, -1),$$

$$\mathbf{o} = \mathbf{x} - \mathbf{p} = (3, 1) - (2, -1) = (1, 2), \quad \|\mathbf{o}\| = \sqrt{5}.$$

Problem. Find the point on the line $y = -x$ that is closest to the point $(3, 4)$.

The required point is the projection $\mathbf{p}$ of $\mathbf{v} = (3, 4)$ on the vector $\mathbf{w} = (1, -1)$ spanning the line $y = -x$.

$$\mathbf{p} = \frac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}} \mathbf{w} = \frac{-1}{2} (1, -1) = \left(-\frac{1}{2}, \frac{1}{2} \right).$$

Problem. Let Π be the plane spanned by vectors $\mathbf{v}_1 = (1,1,0)$ and $\mathbf{v}_2 = (0,1,1)$.

(i) Find the orthogonal projection of the vector $\mathbf{x} = (4,0,-1)$ onto the plane Π.

(ii) Find the distance from $\mathbf{x}$ to Π.

We have $\mathbf{x} = \mathbf{p} + \mathbf{o}$, where $\mathbf{p} \in \Pi$ and $\mathbf{o} \perp \Pi$. Then the orthogonal projection of $\mathbf{x}$ onto Π is $\mathbf{p}$ and the distance from $\mathbf{x}$ to Π is $\|\mathbf{o}\|$.

We have $\mathbf{p} = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2$ for some $\alpha, \beta \in \mathbb{R}$. Then $\mathbf{o} = \mathbf{x} - \mathbf{p} = \mathbf{x} - \alpha\mathbf{v}_1 - \beta\mathbf{v}_2$.

$$\begin{cases} \mathbf{o} \cdot \mathbf{v}_1 = 0 \\ \mathbf{o} \cdot \mathbf{v}_2 = 0 \end{cases} \iff \begin{cases} \alpha(\mathbf{v}_1 \cdot \mathbf{v}_1) + \beta(\mathbf{v}_2 \cdot \mathbf{v}_1) = \mathbf{x} \cdot \mathbf{v}_1 \\ \alpha(\mathbf{v}_1 \cdot \mathbf{v}_2) + \beta(\mathbf{v}_2 \cdot \mathbf{v}_2) = \mathbf{x} \cdot \mathbf{v}_2 \end{cases}$$

$$\mathbf{x} = (4, 0, -1), \ \ \mathbf{v}_1 = (1, 1, 0), \ \ \mathbf{v}_2 = (0, 1, 1)$$

$$\begin{cases} \alpha(\mathbf{v}_1 \cdot \mathbf{v}_1) + \beta(\mathbf{v}_2 \cdot \mathbf{v}_1) = \mathbf{x} \cdot \mathbf{v}_1 \\ \alpha(\mathbf{v}_1 \cdot \mathbf{v}_2) + \beta(\mathbf{v}_2 \cdot \mathbf{v}_2) = \mathbf{x} \cdot \mathbf{v}_2 \end{cases}$$

$$\iff \begin{cases} 2\alpha + \beta = 4 \\ \alpha + 2\beta = -1 \end{cases} \iff \begin{cases} \alpha = 3 \\ \beta = -2 \end{cases}$$

$$\mathbf{p} = 3\mathbf{v}_1 - 2\mathbf{v}_2 = (3, 1, -2)$$

$$\mathbf{o} = \mathbf{x} - \mathbf{p} = (1, -1, 1)$$

$$\|\mathbf{o}\| = \sqrt{3}$$

Problem. Let Π be the plane spanned by vectors $\mathbf{v}_1 = (1,1,0)$ and $\mathbf{v}_2 = (0,1,1)$.

(i) Find the orthogonal projection of the vector $\mathbf{x} = (4,0,-1)$ onto the plane Π.

(ii) Find the distance from $\mathbf{x}$ to Π.

Alternative solution: We have $\mathbf{x} = \mathbf{p} + \mathbf{o}$, where $\mathbf{p} \in \Pi$ and $\mathbf{o} \perp \Pi$. Then the orthogonal projection of $\mathbf{x}$ onto Π is $\mathbf{p}$ and the distance from $\mathbf{x}$ to Π is $\|\mathbf{o}\|$.

Notice that $\mathbf{o}$ is the orthogonal projection of $\mathbf{x}$ onto the orthogonal complement $\Pi^\perp$. In the previous lecture, we found that $\Pi^\perp$ is the line spanned by the vector $\mathbf{y} = (1,-1,1)$. It follows that

$$\mathbf{o} = \frac{\mathbf{x}\cdot\mathbf{y}}{\mathbf{y}\cdot\mathbf{y}}\,\mathbf{y} = \frac{3}{3}\,(1,-1,1) = (1,-1,1).$$

Then $\mathbf{p} = \mathbf{x} - \mathbf{o} = (4,0,-1) - (1,-1,1) = (3,1,-2)$ and $\|\mathbf{o}\| = \sqrt{3}$.

Overdetermined system of linear equations:

$$\begin{cases} x + 2y = 3 \\ 3x + 2y = 5 \\ x + y = 2.09 \end{cases} \iff \begin{cases} x + 2y = 3 \\ -4y = -4 \\ -y = -0.91 \end{cases}$$

No solution: inconsistent system

Assume that a solution (x_0, y_0) does exist but the system is not quite accurate, namely, there may be some errors in the right-hand sides.

Problem. Find a good approximation of (x_0, y_0).

One approach is the **least squares fit**. Namely, we look for a pair (x, y) that minimizes the sum $(x + 2y - 3)^2 + (3x + 2y - 5)^2 + (x + y - 2.09)^2$.

Least squares solution

System of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \iff A\mathbf{x} = \mathbf{b}$$

For any $\mathbf{x} \in \mathbb{R}^n$ define a **residual** $r(\mathbf{x}) = \mathbf{b} - A\mathbf{x}$.

The **least squares solution x** to the system is the one that minimizes $\|r(\mathbf{x})\|$ (or, equivalently, $\|r(\mathbf{x})\|^2$).

$$\|r(\mathbf{x})\|^2 = \sum_{i=1}^{m} (a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n - b_i)^2$$

Let A be an $m \times n$ matrix and let $\mathbf{b} \in \mathbb{R}^m$.

Theorem A vector $\hat{\mathbf{x}}$ is a least squares solution of the system $A\mathbf{x} = \mathbf{b}$ if and only if it is a solution of the associated **normal system** $\boxed{A^T A\mathbf{x} = A^T\mathbf{b}.}$

Proof: $A\mathbf{x}$ is an arbitrary vector in $R(A)$, the column space of A. Hence the length of $r(\mathbf{x}) = \mathbf{b} - A\mathbf{x}$ is minimal if $A\mathbf{x}$ is the orthogonal projection of $\mathbf{b}$ onto $R(A)$. That is, if $r(\mathbf{x})$ is orthogonal to $R(A)$.

We know that $R(A)^\perp = N(A^T)$, the nullspace of the transpose matrix. Thus $\hat{\mathbf{x}}$ is a least squares solution if and only if

$$A^T r(\hat{\mathbf{x}}) = \mathbf{0} \iff A^T(\mathbf{b} - A\hat{\mathbf{x}}) = \mathbf{0} \iff A^T A\hat{\mathbf{x}} = A^T\mathbf{b}.$$

Problem. Find the least squares solution to

$$\begin{cases} x + 2y = 3 \\ 3x + 2y = 5 \\ x + y = 2.09 \end{cases}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 2.09 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \\ 2.09 \end{pmatrix}$$

$$\begin{pmatrix} 11 & 9 \\ 9 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 20.09 \\ 18.09 \end{pmatrix} \iff \begin{cases} x = 1 \\ y = 1.01 \end{cases}$$