MATH 323–502 Fall 2022

Homework assignment #11

Problem 1. Consider the inner product space $C[0,1]$ with the inner product defined by

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$$

and the induced norm. Find the angle θ between the functions $h_1(x) = 1$ and $h_2(x) = x$.

Problem 2. Sketch the set of points $\mathbf{x} = (x_1, x_2)$ in $\mathbb{R}^2$ such that

(i) $\|\mathbf{x}\|_2 = 1$, **(ii)** $\|\mathbf{x}\|_1 = 1$, **(iii)** $\|\mathbf{x}\|_\infty = 1$.

Problem 3. Suppose $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis for an inner product space V. Find the angle θ between the vectors $\mathbf{u} = \mathbf{u}_1 + 2\mathbf{u}_2 + 2\mathbf{u}_3$ and $\mathbf{v} = \mathbf{u}_1 + 7\mathbf{u}_3$.

Problem 4. Consider the inner product space $C[-1,1]$ with the inner product defined by

$$\langle f, g\rangle = \int_{-1}^1 f(x)g(x)\,dx$$

and the induced norm. Find the best least squares approximation to the function $f(x) = x^{1/3}$ on $[-1,1]$ by a linear function $\ell(x) = c_1 + c_2 x$.
[Hint: first show that the functions $h_1(x) = 1$ and $h_2(x) = x$ are orthogonal.]

Problem 5. Consider the inner product space from Problem 1. Let V be the subspace spanned by the functions $h_1(x) = 1$ and $h_2(x) = 2x - 1$. Find the best least squares approximation to the function $f(x) = \sqrt{x}$ on $[0,1]$ by a function from V.
[Hint: first show that h_1 and h_2 are orthogonal.]

Problem 6. Use the Gram-Schmidt process to find an orthonormal basis for the column space of the matrix

$$\begin{pmatrix} -1 & 3 \\ 1 & 5 \end{pmatrix}.$$

Problem 7. Given the basis $\{(1,2,-2), (4,3,2), (1,2,1)\}$ for $\mathbb{R}^3$, use the Gram-Schmidt process to obtain an orthonormal basis.

Problem 8. Consider the inner product space from Problem 4. Find an orthonormal basis for the subspace of $C[-1,1]$ spanned by functions $h_1(x) = 1$, $h_2(x) = x$ and $h_3(x) = x^2$.

Problem 9. Verify that vectors $\mathbf{x}_1 = \frac{1}{2}(1,1,1,-1)$ and $\mathbf{x}_2 = \frac{1}{6}(1,1,3,5)$ form an orthonormal set in $\mathbb{R}^4$. Extend this set to an orthonormal basis for $\mathbb{R}^4$.
[Hint: first find a basis for the orthogonal complement of the subspace spanned by $\mathbf{x}_1$ and $\mathbf{x}_2$ and then use the Gram-Schmidt process.]

Problem 10. Use the Gram-Schmidt process to find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by vectors $\mathbf{x}_1 = (4,2,2,1)$, $\mathbf{x}_2 = (2,0,0,2)$ and $\mathbf{x}_3 = (1,1,-1,1)$.