

MATH 323
Linear Algebra

Lecture 2:
Gaussian elimination.
Gauss-Jordan reduction.

System of linear equations

$$\left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array} \right.$$

Here x_1, x_2, \dots, x_n are variables and a_{ij}, b_j are constants.

A *solution* of the system is a common solution of all equations in the system.

A system of linear equations can have **one** solution, **infinitely many** solutions, or **no** solution at all.

Solving systems of linear equations

Elimination method always works for systems of linear equations.

Algorithm: (1) pick a variable, solve one of the equations for it, and eliminate it from the other equations; (2) put aside the equation used in the elimination, and return to step (1).

$$x - y = -2 \implies x = y - 2$$

$$2x + 3y = 6 \implies 2(y - 2) + 3y = 6$$

After the elimination is completed, the system is solved by *back substitution*.

$$y = 2 \implies x = y - 2 = 0$$

Example.

$$\begin{cases} x - y & = 2 \\ 2x - y - z & = 3 \\ x + y + z & = 6 \end{cases}$$

After elimination:

$$\begin{cases} x = y + 2 \\ y = z - 1 \\ 3z = 6 \end{cases}$$

After back substitution:

$$\begin{cases} x = 3 \\ y = 1 \\ z = 2 \end{cases}$$

Another example.

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 14 \end{cases}$$

After elimination:

$$\begin{cases} x = -y + 2z + 1 \\ y = z + 3 \\ 0 = 0 \end{cases}$$

After back substitution:

$$\begin{cases} x = t - 2 \\ y = t + 3 \\ z = t \end{cases} \quad \text{for any } t \in \mathbb{R}.$$

Gaussian elimination

Gaussian elimination is a modification of the elimination method that allows only so-called *elementary operations*.

Elementary operations for systems of linear equations:

- (1) to multiply an equation by a nonzero scalar;
- (2) to add an equation multiplied by a scalar to another equation;
- (3) to interchange two equations.

Theorem (i) Applying elementary operations to a system of linear equations does not change the solution set of the system. **(ii)** Any elementary operation can be undone by another elementary operation.

Operation 1: multiply the i th equation by $r \neq 0$.

$$\begin{aligned} & \left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \dots\dots\dots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \\ \dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array} \right. \\ \implies & \left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \dots\dots\dots \\ (ra_{i1})x_1 + (ra_{i2})x_2 + \cdots + (ra_{in})x_n = rb_i \\ \dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array} \right. \end{aligned}$$

To undo the operation, multiply the i th equation by r^{-1} .

Operation 2: add r times the i th equation to the j th equation.

$$\left\{ \begin{array}{c} \dots\dots\dots \\ a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i \\ \dots\dots\dots \\ a_{j1}x_1 + a_{j2}x_2 + \dots + a_{jn}x_n = b_j \\ \dots\dots\dots \end{array} \right. \implies$$

$$\left\{ \begin{array}{c} \dots\dots\dots \\ a_{i1}x_1 + \dots + a_{in}x_n = b_i \\ \dots\dots\dots \\ (a_{j1} + ra_{i1})x_1 + \dots + (a_{jn} + ra_{in})x_n = b_j + rb_i \\ \dots\dots\dots \end{array} \right.$$

To undo the operation, add $-r$ times the i th equation to the j th equation.

Operation 3: interchange the i th and j th equations.

$$\begin{aligned} & \left\{ \begin{array}{l} \dots\dots\dots \\ a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i \\ \dots\dots\dots \\ a_{j1}x_1 + a_{j2}x_2 + \dots + a_{jn}x_n = b_j \\ \dots\dots\dots \end{array} \right. \\ \Rightarrow & \left\{ \begin{array}{l} \dots\dots\dots \\ a_{j1}x_1 + a_{j2}x_2 + \dots + a_{jn}x_n = b_j \\ \dots\dots\dots \\ a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i \\ \dots\dots\dots \end{array} \right. \end{aligned}$$

To undo the operation, apply it once more.

Example.

$$\begin{cases} x - y & = 2 \\ 2x - y - z & = 3 \\ x + y + z & = 6 \end{cases}$$

Add -2 times the 1st equation to the 2nd equation:

$$\begin{cases} x - y & = 2 \\ y - z & = -1 \\ x + y + z & = 6 \end{cases} \quad \boxed{R_2 := R_2 - 2 * R_1}$$

Add -1 times the 1st equation to the 3rd equation:

$$\begin{cases} x - y & = 2 \\ y - z & = -1 \\ 2y + z & = 4 \end{cases}$$

Add -2 times the 2nd equation to the 3rd equation:

$$\begin{cases} x - y & = & 2 \\ & y - z & = & -1 \\ & & 3z & = & 6 \end{cases}$$

The elimination is completed, and we can solve the system by back substitution. However we can as well proceed with elementary operations.

Multiply the 3rd equation by $1/3$:

$$\begin{cases} x - y & = & 2 \\ & y - z & = & -1 \\ & & z & = & 2 \end{cases}$$

Add the 3rd equation to the 2nd equation:

$$\begin{cases} x - y & = 2 \\ y & = 1 \\ z & = 2 \end{cases}$$

Add the 2nd equation to the 1st equation:

$$\begin{cases} x & = 3 \\ y & = 1 \\ z & = 2 \end{cases}$$

System of linear equations:

$$\begin{cases} x - y & = 2 \\ 2x - y - z & = 3 \\ x + y + z & = 6 \end{cases}$$

Solution: $(x, y, z) = (3, 1, 2)$

Another example.

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 14 \end{cases}$$

Add the 1st equation to the 3rd equation:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ 5y - 5z = 15 \end{cases}$$

Add -5 times the 2nd equation to the 3rd equation:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ 0 = 0 \end{cases}$$

Add -1 times the 2nd equation to the 1st equation:

$$\begin{cases} x & - z = -2 \\ & y - z = 3 \\ & 0 = 0 \end{cases} \iff \begin{cases} x = z - 2 \\ y = z + 3 \end{cases}$$

Here z is a *free variable* (x and y are *leading variables*).

It follows that
$$\begin{cases} x = t - 2 \\ y = t + 3 \\ z = t \end{cases} \quad \text{for any } t \in \mathbb{R}.$$

System of linear equations:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 14 \end{cases}$$

Solution: $(x, y, z) = (t - 2, t + 3, t), \quad t \in \mathbb{R}.$

In vector form, $(x, y, z) = (-2, 3, 0) + t(1, 1, 1).$

The set of all solutions is a straight line in \mathbb{R}^3 passing through the point $(-2, 3, 0)$ in the direction $(1, 1, 1).$

Yet another example.

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 1 \end{cases}$$

Add the 1st equation to the 3rd equation:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ 5y - 5z = 2 \end{cases}$$

Add -5 times the 2nd equation to the 3rd equation:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ 0 = -13 \end{cases}$$

System of linear equations:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 1 \end{cases}$$

Solution: no solution (*inconsistent system*).

Matrices

Definition. A *matrix* is a rectangular array of numbers.

Examples: $\begin{pmatrix} 2 & 7 \\ -1 & 0 \\ 3 & 3 \end{pmatrix}$, $\begin{pmatrix} 2 & 7 & 0.2 \\ 4.6 & 1 & 1 \end{pmatrix}$,

$\begin{pmatrix} 3/5 \\ 5/8 \\ 4 \end{pmatrix}$, $(\sqrt{2}, 0, -\sqrt{3}, 5)$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

dimensions = (# of rows) \times (# of columns)

$n \times n$: **square matrix**

$n \times 1$: **column vector**

$1 \times n$: **row vector**

System of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Coefficient matrix and column vector of the right-hand sides:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

System of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Augmented matrix:

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array} \right)$$

Since the elementary operations preserve the standard form of linear equations, we can trace the solution process by looking on the augmented matrix.

Elementary operations for systems of linear equations correspond to *elementary row operations* for augmented matrices:

- (1) to multiply a row by a nonzero scalar;
- (2) to add the i th row multiplied by some $r \in \mathbb{R}$ to the j th row;
- (3) to interchange two rows.

Remark. Rows are added and multiplied by scalars as vectors (namely, row vectors).

Elementary row operations

Augmented matrix:

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array} \right) = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{pmatrix},$$

where $\mathbf{v}_i = (a_{i1} \ a_{i2} \ \dots \ a_{in} \mid b_i)$ is a row vector.

Elementary row operations

Operation 1: to multiply the i th row by $r \neq 0$:

$$\begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix} \rightarrow \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ r\mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix}$$

Elementary row operations

Operation 2: to add the i th row multiplied by r to the j th row:

$$\begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_j \\ \vdots \\ \mathbf{v}_m \end{pmatrix} \rightarrow \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_j + r\mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix}$$

Elementary row operations

Operation 3: to interchange the i th row with the j th row:

$$\begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_j \\ \vdots \\ \mathbf{v}_m \end{pmatrix} \rightarrow \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_j \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix}$$

Row echelon form

General augmented matrix in row echelon form:

$$\left(\begin{array}{cccccccccccc|c} \boxed{} & * & * & * & * & * & * & * & * & * & * & * \\ & \boxed{} & \circledast & \circledast & * & * & * & * & * & * & * & * \\ & & & & \boxed{} & \circledast & * & * & * & * & * & * \\ & & & & & & \boxed{} & * & * & * & * & * \\ & & & & & & & \boxed{} & * & * & * & * \\ & & & & & & & & \boxed{} & \circledast & \circledast & * \end{array} \right)$$

- leading entries are boxed (all equal to 1);
- all the entries below the staircase line are zero;
- each step of the staircase has height 1;
- each circle marks a column without a leading entry (that corresponds to a free variable).

Strict triangular form is a particular case of row echelon form that can occur for systems of n equations in n variables:

$$\left(\begin{array}{cccccccc|c} \square & * & * & * & * & * & * & * & * \\ & \square & * & * & * & * & * & * & * \\ & & \square & * & * & * & * & * & * \\ & & & \square & * & * & * & * & * \\ & & & & \square & * & * & * & * \\ & & & & & \square & * & * & * \\ & & & & & & \square & * & * \\ & & & & & & & \square & * \end{array} \right)$$

- no zero rows;
- no free variables.

Consistency check

The original system of linear equations is **consistent** if there is no leading entry in the rightmost column of the augmented matrix in row echelon form.

The diagram shows an augmented matrix in row echelon form. The matrix is enclosed in large parentheses. A vertical line separates the coefficient matrix from the rightmost column, which contains the constants. A blue staircase line indicates the leading entries in each row. The leading entries are represented by squares. The rightmost column contains asterisks. The asterisks in the second, third, and sixth rows from the top are circled in red, indicating that there are leading ones in the rightmost column, which signifies an inconsistent system.

$$\left(\begin{array}{cccccccccc|c} \square & * & * & * & * & * & * & * & * & * & * \\ \square & * & * & * & * & * & * & * & * & * & * \\ \square & * & * & * & * & * & * & * & * & * & * \\ \square & * & * & * & * & * & * & * & * & * & * \\ \square & * & * & * & * & * & * & * & * & * & * \\ \square & * & * & * & * & * & * & * & * & * & * \\ \square & * & * & * & * & * & * & * & * & * & * \\ \square & * & * & * & * & * & * & * & * & * & * \\ \square & * & * & * & * & * & * & * & * & * & * \\ \square & * & * & * & * & * & * & * & * & * & * \end{array} \right)$$

Augmented matrix of an inconsistent system

The goal of the **Gauss-Jordan reduction** is to convert the augmented matrix into **reduced row echelon form**:

$$\left(\begin{array}{cccc|ccc} \boxed{1} & * & * & * & * & * & * \\ & \boxed{1} & \circledast & \circledast & * & * & * \\ & & & \boxed{1} & \circledast & * & * \\ & & & & \boxed{1} & * & * \\ & & & & & \boxed{1} & \circledast & \circledast & * \end{array} \right)$$

- all entries below the staircase line are zero;
- each boxed entry is 1, the other entries in its column are zero;
- each circle corresponds to a free variable.

Example.

$$\begin{cases} x - y & = 2 \\ 2x - y - z & = 3 \\ x + y + z & = 6 \end{cases} \quad \left(\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 2 & -1 & -1 & 3 \\ 1 & 1 & 1 & 6 \end{array} \right)$$

Row echelon form (also strict triangular):

$$\begin{cases} x - y & = 2 \\ y - z & = -1 \\ z & = 2 \end{cases} \quad \left(\begin{array}{ccc|c} \boxed{1} & -1 & 0 & 2 \\ 0 & \boxed{1} & -1 & -1 \\ 0 & 0 & \boxed{1} & 2 \end{array} \right)$$

Reduced row echelon form:

$$\begin{cases} x & = 3 \\ y & = 1 \\ z & = 2 \end{cases} \quad \left(\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 3 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & \boxed{1} & 2 \end{array} \right)$$

Another example.

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 14 \end{cases} \quad \left(\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 3 \\ -1 & 4 & -3 & 14 \end{array} \right)$$

Row echelon form:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ 0 = 0 \end{cases} \quad \left(\begin{array}{ccc|c} \boxed{1} & 1 & -2 & 1 \\ 0 & \boxed{1} & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

Reduced row echelon form:

$$\begin{cases} x - z = -2 \\ y - z = 3 \\ 0 = 0 \end{cases} \quad \left(\begin{array}{ccc|c} \boxed{1} & 0 & -1 & -2 \\ 0 & \boxed{1} & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

Yet another example.

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 1 \end{cases} \quad \left(\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 3 \\ -1 & 4 & -3 & 1 \end{array} \right)$$

Row echelon form:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ 0 = 1 \end{cases} \quad \left(\begin{array}{ccc|c} \boxed{1} & 1 & -2 & 1 \\ 0 & \boxed{1} & -1 & 3 \\ 0 & 0 & 0 & \boxed{1} \end{array} \right)$$

Reduced row echelon form:

$$\begin{cases} x - z = 0 \\ y - z = 0 \\ 0 = 1 \end{cases} \quad \left(\begin{array}{ccc|c} \boxed{1} & 0 & -1 & 0 \\ 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{array} \right)$$

How to solve a system of linear equations

- Order the variables.
- Write down the augmented matrix of the system.
- Convert the matrix to **row echelon form**.
- Check for consistency.
- Convert the matrix to **reduced row echelon form**.
- Write down the system corresponding to the reduced row echelon form.
- Determine leading and free variables.
- Rewrite the system so that the leading variables are on the left while everything else is on the right.
- Assign parameters to the free variables and write down the general solution in parametric form.

New example.
$$\begin{cases} x_2 + 2x_3 + 3x_4 = 6 \\ x_1 + 2x_2 + 3x_3 + 4x_4 = 10 \end{cases}$$

Variables: x_1, x_2, x_3, x_4 .

Augmented matrix:
$$\left(\begin{array}{cccc|c} 0 & 1 & 2 & 3 & 6 \\ 1 & 2 & 3 & 4 & 10 \end{array} \right)$$

To get it into row echelon form, we exchange the two rows:

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 10 \\ 0 & 1 & 2 & 3 & 6 \end{array} \right)$$

Consistency check is passed. To convert into reduced row echelon form, add -2 times the 2nd row to the 1st row:

$$\left(\begin{array}{cccc|c} \boxed{1} & 0 & -1 & -2 & -2 \\ 0 & \boxed{1} & 2 & 3 & 6 \end{array} \right)$$

The leading variables are x_1 and x_2 ; hence x_3 and x_4 are free variables.

Back to the system:

$$\begin{cases} x_1 - x_3 - 2x_4 = -2 \\ x_2 + 2x_3 + 3x_4 = 6 \end{cases} \iff \begin{cases} x_1 = x_3 + 2x_4 - 2 \\ x_2 = -2x_3 - 3x_4 + 6 \end{cases}$$

General solution:

$$\begin{cases} x_1 = t + 2s - 2 \\ x_2 = -2t - 3s + 6 \\ x_3 = t \\ x_4 = s \end{cases} \quad (t, s \in \mathbb{R})$$

In vector form, $(x_1, x_2, x_3, x_4) =$
 $= (-2, 6, 0, 0) + t(1, -2, 1, 0) + s(2, -3, 0, 1).$