

MATH 323
Linear Algebra

Lecture 7:
Determinants (continued).

Determinants

Determinant is a scalar assigned to each square matrix.

Notation. The determinant of a matrix

$A = (a_{ij})_{1 \leq i, j \leq n}$ is denoted $\det A$ or

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

Principal property: $\det A \neq 0$ if and only if a system of linear equations with the coefficient matrix A has a unique solution. Equivalently, $\det A \neq 0$ if and only if the matrix A is invertible.

Explicit definition in low dimensions

Definition. $\det(a) = a$, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

$$+ : \begin{pmatrix} \boxed{*} & * & * \\ * & \boxed{*} & * \\ * & * & \boxed{*} \end{pmatrix}, \begin{pmatrix} * & \boxed{*} & * \\ * & * & \boxed{*} \\ \boxed{*} & * & * \end{pmatrix}, \begin{pmatrix} * & * & \boxed{*} \\ \boxed{*} & * & * \\ * & \boxed{*} & * \end{pmatrix}.$$

$$- : \begin{pmatrix} * & * & \boxed{*} \\ * & \boxed{*} & * \\ \boxed{*} & * & * \end{pmatrix}, \begin{pmatrix} * & \boxed{*} & * \\ \boxed{*} & * & * \\ * & * & \boxed{*} \end{pmatrix}, \begin{pmatrix} \boxed{*} & * & * \\ * & * & \boxed{*} \\ * & \boxed{*} & * \end{pmatrix}.$$

General definition

The general definition of the determinant is quite complicated as there is no simple explicit formula.

There are several approaches to defining determinants.

Approach 1 (original): an explicit (but very complicated) formula.

Approach 2 (axiomatic): we formulate properties that the determinant should have.

Approach 3 (inductive): the determinant of an $n \times n$ matrix is defined in terms of determinants of certain $(n - 1) \times (n - 1)$ matrices.

Classical definition

Definition. If $A = (a_{ij})$ is an $n \times n$ matrix then

$$\det A = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) a_{1,\pi(1)} a_{2,\pi(2)} \cdots a_{n,\pi(n)},$$

where π runs over S_n , the set of all permutations of $\{1, 2, \dots, n\}$, and $\operatorname{sgn}(\pi)$ denotes the sign of the permutation π .

Remarks. • A **permutation** of the set $\{1, 2, \dots, n\}$ is an invertible mapping of this set onto itself. There are $n!$ such mappings.

• The **sign** $\operatorname{sgn}(\pi)$ can be 1 or -1 . Its definition is rather complicated.

Axiomatic definition

$\mathcal{M}_{n,n}(\mathbb{R})$: the set of $n \times n$ matrices with real entries.

Theorem There exists a unique function $\det : \mathcal{M}_{n,n}(\mathbb{R}) \rightarrow \mathbb{R}$ (called the determinant) with the following properties:

(D1) if a row of a matrix is multiplied by a scalar r , the determinant is also multiplied by r ;

(D2) if we add a row of a matrix multiplied by a scalar to another row, the determinant remains the same;

(D3) if we interchange two rows of a matrix, the determinant changes its sign;

(D4) $\det I = 1$.

Corollary 1 Suppose A is a square matrix and B is obtained from A applying elementary row operations. Then $\det A = 0$ if and only if $\det B = 0$.

Corollary 2 $\det B = 0$ whenever the matrix B has a zero row.

Hint: Multiply the zero row by the zero scalar.

Corollary 3 $\det A = 0$ if and only if the matrix A is not invertible.

Idea of the proof: Let B be the reduced row echelon form of A . If A is invertible then $B = I$; otherwise B has a zero row.

Remark. The same argument proves that properties (D1)–(D4) are enough to evaluate any determinant.

Row echelon form of a square matrix A :

A 7x7 matrix in row echelon form. The diagonal elements are represented by squares, and all other elements are asterisks. A blue line connects the bottom-left corners of the squares, forming a staircase pattern that descends from the top-left to the bottom-right.

$$\begin{pmatrix} \square & * & * & * & * & * & * \\ & \square & * & * & * & * & * \\ & & \square & * & * & * & * \\ & & & \square & * & * & * \\ & & & & \square & * & * \\ & & & & & \square & * \\ & & & & & & \square \end{pmatrix}$$

$$\det A \neq 0$$

A 7x7 matrix in row echelon form, similar to the first one. However, the asterisks in the third row, fourth column and the sixth row, seventh column are circled in red. A blue line connects the bottom-left corners of the squares, forming a staircase pattern that descends from the top-left to the bottom-right.

$$\begin{pmatrix} \square & * & * & * & * & * & * \\ & \square & * & * & * & * & * \\ & & \square & * & * & * & * \\ & & & \square & * & * & * \\ & & & & \square & * & * \\ & & & & & \square & * \\ & & & & & & \square \end{pmatrix}$$

$$\det A = 0$$

Example. $A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}$, $\det A = ?$

Earlier we have transformed the matrix A into the identity matrix using elementary row operations:

- interchange the 1st row with the 2nd row,
- add -3 times the 1st row to the 2nd row,
- add 2 times the 1st row to the 3rd row,
- multiply the 2nd row by -0.5 ,
- add -3 times the 2nd row to the 3rd row,
- multiply the 3rd row by -0.4 ,
- add -1.5 times the 3rd row to the 2nd row,
- add -1 times the 3rd row to the 1st row.

Example. $A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}$, $\det A = ?$

Earlier we have transformed the matrix A into the identity matrix using elementary row operations.

These included two row multiplications, by -0.5 and by -0.4 , and one row exchange.

It follows that

$$\det I = -(-0.5)(-0.4) \det A = (-0.2) \det A.$$

Hence $\det A = -5 \det I = -5$.

Tests for singularity

- If a matrix A has a zero row then $\det A = 0$.
- If a matrix A has two identical rows then $\det A = 0$.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & 0 \end{vmatrix} = 0.$$

- If a matrix A has two proportional rows then $\det A = 0$.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ ra_1 & ra_2 & ra_3 \end{vmatrix} = r \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = 0.$$

Additive law for rows

- Suppose that matrices X, Y, Z are identical except for the i th row and the i th row of Z is the sum of the i th rows of X and Y .

Then $\boxed{\det Z = \det X + \det Y.}$

$$\begin{vmatrix} a_1+a'_1 & a_2+a'_2 & a_3+a'_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a'_1 & a'_2 & a'_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

- Adding a scalar multiple of one row to another row does not change the determinant of a matrix.

$$\begin{aligned} & \begin{vmatrix} a_1 + rb_1 & a_2 + rb_2 & a_3 + rb_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \\ & = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} rb_1 & rb_2 & rb_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \end{aligned}$$

Special matrices

- $\det I = 1$.
- If $A = \text{diag}(d_1, d_2, \dots, d_n)$ then $\det A = d_1 d_2 \dots d_n$.

Definition. A square matrix $A = (a_{ij})$ is called **upper triangular** if all entries below the main diagonal are zeros: $a_{ij} = 0$ whenever $i > j$.

- The determinant of an upper triangular matrix is equal to the product of its diagonal entries.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33}.$$

Determinant of the transpose

- If A is a square matrix then $\det A^T = \det A$.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

As a consequence, for every property of determinants involving rows of a matrix there is an analogous property involving columns of a matrix.

Columns vs. rows

- If one column of a matrix is multiplied by a scalar, the determinant is multiplied by the same scalar.
- Interchanging two columns of a matrix changes the sign of its determinant.
- If a matrix A has two columns proportional then $\det A = 0$.
- Adding a scalar multiple of one column to another does not change the determinant of a matrix.

Submatrices

Definition. Given a matrix A , a $k \times k$ **submatrix** of A is a matrix obtained by specifying k columns and k rows of A and deleting the other columns and rows.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 10 & 20 & 30 & 40 \\ 3 & 5 & 7 & 9 \end{pmatrix} \rightarrow \begin{pmatrix} * & 2 & * & 4 \\ * & * & * & * \\ * & 5 & * & 9 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 4 \\ 5 & 9 \end{pmatrix}$$

Row and column expansions

Given an $n \times n$ matrix $A = (a_{ij})$, let M_{ij} denote the $(n-1) \times (n-1)$ submatrix obtained by deleting the i th row and the j th column of A .

Theorem For any $1 \leq k, m \leq n$ we have that

$$\det A = \sum_{j=1}^n (-1)^{k+j} a_{kj} \det M_{kj},$$

(expansion by k th row)

$$\det A = \sum_{i=1}^n (-1)^{i+m} a_{im} \det M_{im}.$$

(expansion by m th column)

Signs for row/column expansions

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Example. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$

Expansion by the 1st row:

$$\begin{pmatrix} \boxed{1} & * & * \\ * & 5 & 6 \\ * & 8 & 9 \end{pmatrix} \quad \begin{pmatrix} * & \boxed{2} & * \\ 4 & * & 6 \\ 7 & * & 9 \end{pmatrix} \quad \begin{pmatrix} * & * & \boxed{3} \\ 4 & 5 & * \\ 7 & 8 & * \end{pmatrix}$$

$$\begin{aligned} \det A &= 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \\ &= (5 \cdot 9 - 6 \cdot 8) - 2(4 \cdot 9 - 6 \cdot 7) + 3(4 \cdot 8 - 5 \cdot 7) = 0. \end{aligned}$$

Example. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$

Expansion by the 2nd column:

$$\begin{pmatrix} * & \boxed{2} & * \\ 4 & * & 6 \\ 7 & * & 9 \end{pmatrix} \quad \begin{pmatrix} 1 & * & 3 \\ * & \boxed{5} & * \\ 7 & * & 9 \end{pmatrix} \quad \begin{pmatrix} 1 & * & 3 \\ 4 & * & 6 \\ * & \boxed{8} & * \end{pmatrix}$$

$$\det A = -2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 5 \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} - 8 \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}$$

$$= -2(4 \cdot 9 - 6 \cdot 7) + 5(1 \cdot 9 - 3 \cdot 7) - 8(1 \cdot 6 - 3 \cdot 4) = 0.$$

Example. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$

Subtract the 1st row from the 2nd row and from the 3rd row:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 6 & 6 & 6 \end{vmatrix} = 0$$

since the last matrix has two proportional rows.

Evaluation of determinants

Example. $B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 13 \end{pmatrix}$.

First let's do some row reduction.

Add -4 times the 1st row to the 2nd row:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 13 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 13 \end{vmatrix}$$

Add -7 times the 1st row to the 3rd row:

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 13 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -8 \end{vmatrix}$$

Expand the determinant by the 1st column:

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -8 \end{vmatrix} = 1 \begin{vmatrix} -3 & -6 \\ -6 & -8 \end{vmatrix}$$

Thus

$$\begin{aligned} \det B &= \begin{vmatrix} -3 & -6 \\ -6 & -8 \end{vmatrix} = (-3) \begin{vmatrix} 1 & 2 \\ -6 & -8 \end{vmatrix} \\ &= (-3)(-2) \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (-3)(-2)(-2) = -12. \end{aligned}$$

Example. $C = \begin{pmatrix} 2 & -2 & 0 & 3 \\ -5 & 3 & 2 & 1 \\ 1 & -1 & 0 & -3 \\ 2 & 0 & 0 & -1 \end{pmatrix}$, $\det C = ?$

Expand the determinant by the 3rd column:

$$\begin{vmatrix} 2 & -2 & 0 & 3 \\ -5 & 3 & 2 & 1 \\ 1 & -1 & 0 & -3 \\ 2 & 0 & 0 & -1 \end{vmatrix} = -2 \begin{vmatrix} 2 & -2 & 3 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix}$$

Add -2 times the 2nd row to the 1st row:

$$\det C = -2 \begin{vmatrix} 2 & -2 & 3 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix} = -2 \begin{vmatrix} 0 & 0 & 9 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix}$$

Expand the determinant by the 1st row:

$$\det C = -2 \begin{vmatrix} 0 & 0 & 9 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix} = -2 \cdot 9 \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix}$$

Thus

$$\det C = -18 \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} = -18 \cdot 2 = -36.$$

Problem. For what values of a will the following system have a unique solution?

$$\begin{cases} x + 2y + z = 1 \\ -x + 4y + 2z = 2 \\ 2x - 2y + az = 3 \end{cases}$$

The system has a unique solution if and only if the coefficient matrix is invertible.

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 4 & 2 \\ 2 & -2 & a \end{pmatrix}, \quad \det A = ?$$

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 4 & 2 \\ 2 & -2 & a \end{pmatrix}, \quad \det A = ?$$

Add -2 times the 3rd column to the 2nd column:

$$\begin{vmatrix} 1 & 2 & 1 \\ -1 & 4 & 2 \\ 2 & -2 & a \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ -1 & 0 & 2 \\ 2 & -2 - 2a & a \end{vmatrix}$$

Expand the determinant by the 2nd column:

$$\det A = \begin{vmatrix} 1 & 0 & 1 \\ -1 & 0 & 2 \\ 2 & -2 - 2a & a \end{vmatrix} = -(-2 - 2a) \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix}$$

Hence $\det A = -(-2 - 2a) \cdot 3 = 6(1 + a)$.

Thus A is invertible if and only if $a \neq -1$.