MATH 323

Linear Algebra

Lecture 13:
Basis and dimension (continued).
Rank of a matrix.

Basis

Definition. Let V be a vector space. Any linearly independent spanning set for V is called a **basis**.

Theorem A nonempty set $S \subset V$ is a basis for V if and only if any vector $\mathbf{v} \in V$ is *uniquely represented* as a linear combination $\mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k$, where $\mathbf{v}_1, \dots, \mathbf{v}_k$ are distinct vectors from S and $r_1, \dots, r_k \in \mathbb{R}$.

Remark on uniqueness. Expansions $\mathbf{v} = 2\mathbf{v}_1 - \mathbf{v}_2$, $\mathbf{v} = -\mathbf{v}_2 + 2\mathbf{v}_1$, and $\mathbf{v} = 2\mathbf{v}_1 - \mathbf{v}_2 + 0\mathbf{v}_3$ are considered the same.

Dimension

Theorem 1 Any vector space has a basis.

Theorem 2 If a vector space V has a finite basis, then all bases for V are finite and have the same number of elements.

Definition. The **dimension** of a vector space V, denoted $\dim V$, is the number of elements in any of its bases.

Examples. • $\dim \mathbb{R}^n = n$

- $\mathcal{M}_{m,n}(\mathbb{R})$: the space of $m \times n$ matrices; $\dim \mathcal{M}_{m,n} = mn$
- $\mathcal{P}_n$: polynomials of degree less than n; $\dim \mathcal{P}_n = n$
- $\mathcal{P}$: the space of all polynomials; $\dim \mathcal{P} = \infty$
- $\{\mathbf{0}\}$: the trivial vector space; $\dim \{\mathbf{0}\} = 0$

Problem. Find the dimension of the plane $x + 2z = 0$ in $\mathbb{R}^3$.

The general solution of the equation $x + 2z = 0$ is

$$\begin{cases} x = -2s \\ y = t \\ z = s \end{cases} \qquad (t, s \in \mathbb{R})$$

That is, $(x, y, z) = (-2s, t, s) = t(0, 1, 0) + s(-2, 0, 1)$.

Hence the plane is the span of vectors $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = (-2, 0, 1)$. These vectors are linearly independent as they are not parallel.

Thus $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis so that the dimension of the plane is 2.

How to find a basis?

Theorem Let S be a subset of a vector space V. Then the following conditions are equivalent:

(i) S is a linearly independent spanning set for V, i.e., a basis;

(ii) S is a minimal spanning set for V;

(iii) S is a maximal linearly independent subset of V.

"Minimal spanning set" means "remove any element from this set, and it is no longer a spanning set".

"Maximal linearly independent subset" means "add any element of V to this set, and it will become linearly dependent".

Theorem Let V be a vector space. Then

(i) any spanning set for V can be reduced to a minimal spanning set;

(ii) any linearly independent subset of V can be extended to a maximal linearly independent set.

Corollary 1 Any spanning set contains a basis while any linearly independent set is contained in a basis.

Corollary 2 A vector space is finite-dimensional if and only if it is spanned by a finite set.

How to find a basis?

Approach 1. Get a spanning set for the vector space, then reduce this set to a basis dropping one vector at a time.

Proposition Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_k$ be a spanning set for a vector space V. If $\mathbf{v}_0$ is a linear combination of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ then $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is also a spanning set for V.

Indeed, if $\mathbf{v}_0 = r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k$, then

$$t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k =$$
$$= (t_0r_1 + t_1)\mathbf{v}_1 + \cdots + (t_0r_k + t_k)\mathbf{v}_k.$$

How to find a basis?

Approach 2. Build a maximal linearly independent set adding one vector at a time.

If the vector space V is trivial, it has the empty basis. If $V \neq \{\mathbf{0}\}$, pick any vector $\mathbf{v}_1 \neq \mathbf{0}$. If $\mathbf{v}_1$ spans V, it is a basis. Otherwise pick any vector $\mathbf{v}_2 \in V$ that is not in the span of $\mathbf{v}_1$. If $\mathbf{v}_1$ and $\mathbf{v}_2$ span V, they constitute a basis. Otherwise pick any vector $\mathbf{v}_3 \in V$ that is not in the span of $\mathbf{v}_1$ and $\mathbf{v}_2$. And so on...

Modifications. Instead of the empty set, we can start with any linearly independent set (if we are given one). If we are given a spanning set S, it is enough to pick new vectors only in S.

Remark. This inductive procedure works for finite-dimensional vector spaces. There is an analogous procedure for infinite-dimensional spaces (*transfinite induction*).

Vectors $\mathbf{v}_1 = (0,1,0)$ and $\mathbf{v}_2 = (-2,0,1)$ are linearly independent.

Problem. Extend the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to a basis for $\mathbb{R}^3$.

Our task is to find a vector $\mathbf{v}_3$ that is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.
Then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ will be a basis for $\mathbb{R}^3$.

Hint 1. $\mathbf{v}_1$ and $\mathbf{v}_2$ span the plane $x + 2z = 0$.

The vector $\mathbf{v}_3 = (1,1,1)$ does not lie in the plane $x + 2z = 0$, hence it is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$.

Vectors $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = (-2, 0, 1)$ are linearly independent.

Problem. Extend the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to a basis for $\mathbb{R}^3$.

Our task is to find a vector $\mathbf{v}_3$ that is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. Then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ will be a basis for $\mathbb{R}^3$.

Hint 2. Since vectors $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$ form a spanning set for $\mathbb{R}^3$, at least one of them can be chosen as $\mathbf{v}_3$.

Let us check that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1\}$ and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_3\}$ are two bases for $\mathbb{R}^3$:

$$\begin{vmatrix} 0 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 1 \neq 0, \qquad \begin{vmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 2 \neq 0.$$

Problem. Find a basis for the vector space V spanned by vectors $\mathbf{w}_1 = (1, 1, 0)$, $\mathbf{w}_2 = (0, 1, 1)$, $\mathbf{w}_3 = (2, 3, 1)$, and $\mathbf{w}_4 = (1, 1, 1)$.

To pare this spanning set, we need to find a relation of the form $r_1\mathbf{w}_1 + r_2\mathbf{w}_2 + r_3\mathbf{w}_3 + r_4\mathbf{w}_4 = \mathbf{0}$, where $r_i \in \mathbb{R}$ are not all equal to zero. Equivalently,

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

To solve this system of linear equations for r_1, r_2, r_3, r_4, we apply row reduction.

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} \color{red}{1} & 0 & 2 & 1 \\ 0 & \color{red}{1} & 1 & 0 \\ 0 & 0 & 0 & \color{red}{1} \end{pmatrix}$$

$$\to \begin{pmatrix} \color{red}{1} & 0 & 2 & 0 \\ 0 & \color{red}{1} & 1 & 0 \\ 0 & 0 & 0 & \color{red}{1} \end{pmatrix} \qquad \text{(reduced row echelon form)}$$

$$\begin{cases} r_1 + 2r_3 = 0 \\ r_2 + r_3 = 0 \\ r_4 = 0 \end{cases} \iff \begin{cases} r_1 = -2r_3 \\ r_2 = -r_3 \\ r_4 = 0 \end{cases}$$

General solution: $(r_1, r_2, r_3, r_4) = (-2t, -t, t, 0)$, $t \in \mathbb{R}$.

Particular solution: $(r_1, r_2, r_3, r_4) = (2, 1, -1, 0)$.

Problem. Find a basis for the vector space V spanned by vectors $\mathbf{w}_1 = (1, 1, 0)$, $\mathbf{w}_2 = (0, 1, 1)$, $\mathbf{w}_3 = (2, 3, 1)$, and $\mathbf{w}_4 = (1, 1, 1)$.

We have obtained that $2\mathbf{w}_1 + \mathbf{w}_2 - \mathbf{w}_3 = \mathbf{0}$. Hence any of vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ can be dropped. For instance, $V = \mathrm{Span}(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4)$.

Let us check whether vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4$ are linearly independent:

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 \neq 0.$$

They are!!! It follows that $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4\}$ is a basis for V. Also, it follows that $V = \mathbb{R}^3$.

Row space of a matrix

Definition. The **row space** of an $m \times n$ matrix A is the subspace of $\mathbb{R}^n$ spanned by rows of A.

The dimension of the row space is called the **rank** of the matrix A.

Theorem 1 The rank of a matrix A is the maximal number of linearly independent rows in A.

Theorem 2 Elementary row operations do not change the row space of a matrix.

Theorem 3 If a matrix A is in row echelon form, then the nonzero rows of A are linearly independent.

Corollary The rank of a matrix is equal to the number of nonzero rows in its row echelon form.

Theorem Elementary row operations do not change the row space of a matrix.

Proof: Suppose that A and B are $m \times n$ matrices such that B is obtained from A by an elementary row operation. Let $\mathbf{a}_1, \dots, \mathbf{a}_m$ be the rows of A and $\mathbf{b}_1, \dots, \mathbf{b}_m$ be the rows of B. We have to show that $\mathrm{Span}(\mathbf{a}_1, \dots, \mathbf{a}_m) = \mathrm{Span}(\mathbf{b}_1, \dots, \mathbf{b}_m)$.

Observe that any row $\mathbf{b}_i$ of B belongs to $\mathrm{Span}(\mathbf{a}_1, \dots, \mathbf{a}_m)$. Indeed, either $\mathbf{b}_i = \mathbf{a}_j$ for some $1 \le j \le m$, or $\mathbf{b}_i = r\mathbf{a}_i$ for some scalar $r \neq 0$, or $\mathbf{b}_i = \mathbf{a}_i + r\mathbf{a}_j$ for some $j \neq i$ and $r \in \mathbb{R}$.

It follows that $\mathrm{Span}(\mathbf{b}_1, \dots, \mathbf{b}_m) \subset \mathrm{Span}(\mathbf{a}_1, \dots, \mathbf{a}_m)$.

Now the matrix A can also be obtained from B by an elementary row operation. By the above,

$$\mathrm{Span}(\mathbf{a}_1, \dots, \mathbf{a}_m) \subset \mathrm{Span}(\mathbf{b}_1, \dots, \mathbf{b}_m).$$

Problem. Find the rank of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Elementary row operations do not change the row space. Let us convert A to row echelon form:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} \color{red}{1} & 1 & 0 \\ 0 & \color{red}{1} & 1 \\ 0 & 0 & \color{red}{1} \\ 0 & 0 & 0 \end{pmatrix}$$

Vectors $(1, 1, 0)$, $(0, 1, 1)$, and $(0, 0, 1)$ form a basis for the row space of A. Thus the rank of A is 3.

It follows that the row space of A is the entire space $\mathbb{R}^3$.

Problem. Find a basis for the vector space V spanned by vectors $\mathbf{w}_1 = (1, 1, 0)$, $\mathbf{w}_2 = (0, 1, 1)$, $\mathbf{w}_3 = (2, 3, 1)$, and $\mathbf{w}_4 = (1, 1, 1)$.

The vector space V is the row space of a matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

According to the solution of the previous problem, vectors $(1, 1, 0)$, $(0, 1, 1)$, and $(0, 0, 1)$ form a basis for V.

Column space of a matrix

Definition. The **column space** of an $m \times n$ matrix A is the subspace of $\mathbb{R}^m$ spanned by columns of A.

Theorem 1 The column space of a matrix A coincides with the row space of the transpose matrix A^T.

Theorem 2 Elementary row operations do not change linear relations between columns of a matrix.

Theorem 3 Elementary row operations do not change the dimension of the column space of a matrix (however they can change the column space).

Theorem 4 If a matrix is in row echelon form, then the columns with leading entries form a basis for the column space.

Corollary For any matrix, the row space and the column space have the same dimension.