

MATH 323

Linear Algebra

**Lecture 14:**

**Rank of a matrix (continued).**

**Basis and coordinates.**

## Row space of a matrix

*Definition.* The **row space** of an  $m \times n$  matrix  $A$  is the subspace of  $\mathbb{R}^n$  spanned by rows of  $A$ .

The dimension of the row space is called the **rank** of the matrix  $A$ .

**Theorem 1** The rank of a matrix  $A$  is the maximal number of linearly independent rows in  $A$ .

**Theorem 2** Elementary row operations do not change the row space of a matrix.

**Theorem 3** If a matrix  $A$  is in row echelon form, then the nonzero rows of  $A$  are linearly independent.

**Corollary** The rank of a matrix is equal to the number of nonzero rows in its row echelon form.

## Column space of a matrix

*Definition.* The **column space** of an  $m \times n$  matrix  $A$  is the subspace of  $\mathbb{R}^m$  spanned by columns of  $A$ .

**Theorem 1** The column space of a matrix  $A$  coincides with the row space of the transpose matrix  $A^T$ .

**Theorem 2** Elementary row operations do not change linear relations between columns of a matrix.

**Theorem 3** Elementary row operations do not change the dimension of the column space of a matrix (however they can change the column space).

**Theorem 4** If a matrix is in row echelon form, then the columns with leading entries form a basis for the column space.

**Corollary** For any matrix, the row space and the column space have the same dimension.

**Problem.** Find a basis for the column space of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

The column space of  $A$  coincides with the row space of  $A^T$ .  
To find a basis, we convert  $A^T$  to row echelon form:

$$A^T = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Vectors  $(1, 0, 2, 1)$ ,  $(0, 1, 1, 0)$ , and  $(0, 0, 0, 1)$  form a basis for the column space of  $A$ .

**Problem.** Find a basis for the column space of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

*Alternative solution:* We already know from the previous lecture that the rank of  $A$  is 3. It follows that the columns of  $A$  are linearly independent. Therefore these columns form a basis for the column space.

**Problem.** Let  $V$  be a vector space spanned by vectors  $\mathbf{w}_1 = (1, 1, 0)$ ,  $\mathbf{w}_2 = (0, 1, 1)$ ,  $\mathbf{w}_3 = (2, 3, 1)$ , and  $\mathbf{w}_4 = (1, 1, 1)$ . Pare this spanning set to a basis for  $V$ .

*Alternative solution:* The vector space  $V$  is the column space of a matrix

$$B = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

The row echelon form of  $B$  is  $C = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ .

Columns of  $C$  with leading entries (1st, 2nd, and 4th) form a basis for the column space of  $C$ . It follows that the corresponding columns of  $B$  (i.e., 1st, 2nd, and 4th) form a basis for the column space of  $B$ .

Thus  $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4\}$  is a basis for  $V$ .

## Nullspace of a matrix

Let  $A = (a_{ij})$  be an  $m \times n$  matrix.

*Definition.* The **nullspace** of the matrix  $A$ , denoted  $N(A)$ , is the set of all  $n$ -dimensional column vectors  $\mathbf{x}$  such that

$$\mathbf{Ax} = \mathbf{0}.$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The nullspace  $N(A)$  is the solution set of a system of linear homogeneous equations (with  $A$  as the coefficient matrix).

**Theorem**  $N(A)$  is a subspace of the vector space  $\mathbb{R}^n$ .

*Definition.* The dimension of the nullspace  $N(A)$  is called the **nullity** of the matrix  $A$ .

## rank + nullity

**Theorem** The rank of a matrix  $A$  plus the nullity of  $A$  equals the number of columns in  $A$ .

*Sketch of the proof:* The rank of  $A$  equals the number of nonzero rows in the row echelon form, which equals the number of leading entries.

The nullity of  $A$  equals the number of free variables in the corresponding homogeneous system, which equals the number of columns without leading entries in the row echelon form.

Consequently, rank+nullity is the number of all columns in the matrix  $A$ .



**Problem.** Find the nullity of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \end{pmatrix}.$$

Clearly, the rows of  $A$  are linearly independent.

Therefore the rank of  $A$  is 2. Since

$$(\text{rank of } A) + (\text{nullity of } A) = 4,$$

it follows that the nullity of  $A$  is 2.

## Basis and dimension

*Definition.* Let  $V$  be a vector space. A linearly independent spanning set for  $V$  is called a **basis**.

**Theorem** Any vector space  $V$  has a basis. If  $V$  has a finite basis, then all bases for  $V$  are finite and have the same number of elements (called the *dimension* of  $V$ ).

*Example.* Vectors  $\mathbf{e}_1 = (1, 0, 0, \dots, 0, 0)$ ,  $\mathbf{e}_2 = (0, 1, 0, \dots, 0, 0), \dots, \mathbf{e}_n = (0, 0, 0, \dots, 0, 1)$  form a basis for  $\mathbb{R}^n$  (called *standard*) since

$$(x_1, x_2, \dots, x_n) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n.$$

## Basis and coordinates

If  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  is a basis for a vector space  $V$ , then any vector  $\mathbf{v} \in V$  has a unique representation

$$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \dots + x_n\mathbf{v}_n,$$

where  $x_i \in \mathbb{R}$ . The coefficients  $x_1, x_2, \dots, x_n$  are called the **coordinates** of  $\mathbf{v}$  with respect to the ordered basis  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ .

The mapping

$$\text{vector } \mathbf{v} \mapsto \text{its coordinates } (x_1, x_2, \dots, x_n)$$

is a one-to-one correspondence between  $V$  and  $\mathbb{R}^n$ . This correspondence respects linear operations in  $V$  and in  $\mathbb{R}^n$ .

*Examples.* • Coordinates of a vector

$\mathbf{v} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$  relative to the standard basis  $\mathbf{e}_1 = (1, 0, \dots, 0, 0)$ ,  $\mathbf{e}_2 = (0, 1, \dots, 0, 0), \dots$ ,  $\mathbf{e}_n = (0, 0, \dots, 0, 1)$  are  $(x_1, x_2, \dots, x_n)$ .

• Coordinates of a matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_{2,2}(\mathbb{R})$

relative to the basis  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$  are  $(a, c, b, d)$ .

• Coordinates of a polynomial

$p(x) = a_0 + a_1x + \dots + a_{n-1}x^{n-1} \in \mathcal{P}_n$  relative to the basis  $1, x, x^2, \dots, x^{n-1}$  are  $(a_0, a_1, \dots, a_{n-1})$ .

## Weird vector space

Consider the set  $V = \mathbb{R}_+$  of positive numbers with a nonstandard addition and scalar multiplication:

$$\boxed{x \oplus y = xy} \quad \text{for any } x, y \in \mathbb{R}_+.$$

$$\boxed{r \odot x = x^r} \quad \text{for any } x \in \mathbb{R}_+ \text{ and } r \in \mathbb{R}.$$

This is an example of a vector space.

The zero vector in  $V$  is the number 1. To build a basis for  $V$ , we can begin with any number  $v \in V$  different from 1. Let's take  $v = 2$ . The span  $\text{Span}(2)$  consists of all numbers of the form  $r \odot 2 = 2^r$ ,  $r \in \mathbb{R}$ . It is the entire space  $V$ . Hence  $\{2\}$  is a basis for  $V$  so that  $\dim V = 1$ .

The coordinate mapping  $f : V \rightarrow \mathbb{R}$  associated to this basis is given by  $f(2^r) = r$  for all  $r \in \mathbb{R}$ . Equivalently,  $f(x) = \log_2 x$ ,  $x \in V$ . Notice that  $\log_2(x \oplus y) = \log_2 x + \log_2 y$  and  $\log_2(r \odot x) = r \log_2 x$ .

Vectors  $\mathbf{u}_1=(3, 1)$  and  $\mathbf{u}_2=(2, 1)$  form a basis for  $\mathbb{R}^2$ .

**Problem 1.** Find coordinates of the vector  $\mathbf{v} = (7, 4)$  with respect to the basis  $\mathbf{u}_1, \mathbf{u}_2$ .

The desired coordinates  $x, y$  satisfy

$$\mathbf{v} = x\mathbf{u}_1 + y\mathbf{u}_2 \iff \begin{cases} 3x + 2y = 7 \\ x + y = 4 \end{cases} \iff \begin{cases} x = -1 \\ y = 5 \end{cases}$$

**Problem 2.** Find the vector  $\mathbf{w}$  whose coordinates with respect to the basis  $\mathbf{u}_1, \mathbf{u}_2$  are  $(7, 4)$ .

$$\mathbf{w} = 7\mathbf{u}_1 + 4\mathbf{u}_2 = 7(3, 1) + 4(2, 1) = (29, 11)$$