

MATH 323

Linear Algebra

**Lecture 15:**

**Change of basis.**

**Linear transformations.**

## Basis and coordinates

If  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  is a basis for a vector space  $V$ , then any vector  $\mathbf{v} \in V$  has a unique representation

$$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \dots + x_n\mathbf{v}_n,$$

where  $x_i \in \mathbb{R}$ . The coefficients  $x_1, x_2, \dots, x_n$  are called the **coordinates** of  $\mathbf{v}$  with respect to the ordered basis  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ .

The mapping

$$\text{vector } \mathbf{v} \mapsto \text{its coordinates } (x_1, x_2, \dots, x_n)$$

is a one-to-one correspondence between  $V$  and  $\mathbb{R}^n$ . This correspondence respects linear operations in  $V$  and in  $\mathbb{R}^n$ .

## Change of coordinates

Given a vector  $\mathbf{v} \in \mathbb{R}^2$ , let  $(x, y)$  be its standard coordinates, i.e., coordinates with respect to the standard basis  $\mathbf{e}_1 = (1, 0)$ ,  $\mathbf{e}_2 = (0, 1)$ , and let  $(x', y')$  be its coordinates with respect to the basis  $\mathbf{u}_1 = (3, 1)$ ,  $\mathbf{u}_2 = (2, 1)$ .

**Problem.** Find a relation between  $(x, y)$  and  $(x', y')$ .

By definition,  $\mathbf{v} = x\mathbf{e}_1 + y\mathbf{e}_2 = x'\mathbf{u}_1 + y'\mathbf{u}_2$ .

In standard coordinates,

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= x' \begin{pmatrix} 3 \\ 1 \end{pmatrix} + y' \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \\ \implies \begin{pmatrix} x' \\ y' \end{pmatrix} &= \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

## Change of coordinates in $\mathbb{R}^n$

The usual (standard) coordinates of a vector  $\mathbf{v} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$  are coordinates relative to the standard basis  $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ . Let  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$  be another basis for  $\mathbb{R}^n$  and  $(x'_1, x'_2, \dots, x'_n)$  be the coordinates of the same vector  $\mathbf{v}$  with respect to this basis. Then

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ u_{21} & u_{22} & \dots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & \dots & u_{nn} \end{pmatrix} \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix},$$

where the matrix  $U = (u_{ij})$  does not depend on the vector  $\mathbf{v}$ . Namely, columns of  $U$  are coordinates of vectors  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$  with respect to the standard basis.  $U$  is called the **transition matrix** from the basis  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$  to the standard basis  $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ . The inverse matrix  $U^{-1}$  is called the **transition matrix** from  $\mathbf{e}_1, \dots, \mathbf{e}_n$  to  $\mathbf{u}_1, \dots, \mathbf{u}_n$ .

**Problem.** Find coordinates of the vector  $\mathbf{x} = (1, 2, 3)$  with respect to the basis  $\mathbf{u}_1 = (1, 1, 0)$ ,  $\mathbf{u}_2 = (0, 1, 1)$ ,  $\mathbf{u}_3 = (1, 1, 1)$ .

The nonstandard coordinates  $(x', y', z')$  of  $\mathbf{x}$  satisfy

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = U \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},$$

where  $U$  is the transition matrix from the standard basis  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  to the basis  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ .

The transition matrix from  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$  to  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  is

$$U_0 = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = \left( \begin{array}{c|c|c} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{array} \right).$$

The transition matrix from  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  to  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$  is the inverse matrix:  $U = U_0^{-1}$ .

The inverse matrix can be computed using row reduction.

$$(U_0 | I) = \left( \begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right)$$

$$\rightarrow \left( \begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right) \rightarrow \left( \begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array} \right)$$

$$\rightarrow \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array} \right) = (I | U_0^{-1})$$

Thus

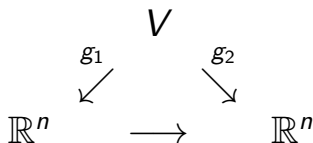
$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$$

## Change of coordinates: general case

Let  $V$  be a vector space of dimension  $n$ .

Let  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  be a basis for  $V$  and  $g_1 : V \rightarrow \mathbb{R}^n$  be the coordinate mapping corresponding to this basis.

Let  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$  be another basis for  $V$  and  $g_2 : V \rightarrow \mathbb{R}^n$  be the coordinate mapping corresponding to this basis.



The composition  $g_2 \circ g_1^{-1}$  is a transformation of  $\mathbb{R}^n$ .

It has the form  $\mathbf{x} \mapsto U\mathbf{x}$ , where  $U$  is an  $n \times n$  matrix.

$U$  is called the **transition matrix** from  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  to  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ . Columns of  $U$  are coordinates of the vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  with respect to the basis  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ .

**Problem.** Find the transition matrix from the basis  $p_1(x) = 1$ ,  $p_2(x) = x + 1$ ,  $p_3(x) = (x + 1)^2$  to the basis  $q_1(x) = 1$ ,  $q_2(x) = x$ ,  $q_3(x) = x^2$  for the vector space  $\mathcal{P}_3$ .

We have to find coordinates of the polynomials  $p_1, p_2, p_3$  with respect to the basis  $q_1, q_2, q_3$ :

$$p_1(x) = 1 = q_1(x),$$

$$p_2(x) = x + 1 = q_1(x) + q_2(x),$$

$$p_3(x) = (x+1)^2 = x^2 + 2x + 1 = q_1(x) + 2q_2(x) + q_3(x).$$

Hence the transition matrix is 
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$



Thus the polynomial identity

$$a_1 + a_2(x + 1) + a_3(x + 1)^2 = b_1 + b_2x + b_3x^2$$

is equivalent to the relation

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

**Problem.** Find the transition matrix from the basis  $\mathbf{v}_1 = (1, 2, 3)$ ,  $\mathbf{v}_2 = (1, 0, 1)$ ,  $\mathbf{v}_3 = (1, 2, 1)$  to the basis  $\mathbf{u}_1 = (1, 1, 0)$ ,  $\mathbf{u}_2 = (0, 1, 1)$ ,  $\mathbf{u}_3 = (1, 1, 1)$ .

It is convenient to make a two-step transition: first from  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  to  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ , and then from  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  to  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ .

Let  $U_1$  be the transition matrix from  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  to  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  and  $U_2$  be the transition matrix from  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$  to  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ :

$$U_1 = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Basis  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \implies$  coordinates  $\mathbf{x}$

Basis  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \implies$  coordinates  $U_1\mathbf{x}$

Basis  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \implies$  coordinates  $U_2^{-1}(U_1\mathbf{x}) = (U_2^{-1}U_1)\mathbf{x}$

Thus the transition matrix from  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  to  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$  is  $U_2^{-1}U_1$ .

$$\begin{aligned} U_2^{-1}U_1 &= \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{pmatrix}. \end{aligned}$$

## Linear mapping = linear transformation = linear function

*Definition.* Given vector spaces  $V_1$  and  $V_2$ , a mapping  $L : V_1 \rightarrow V_2$  is **linear** if

$$L(\mathbf{x} + \mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y}),$$

$$L(r\mathbf{x}) = rL(\mathbf{x})$$

for any  $\mathbf{x}, \mathbf{y} \in V_1$  and  $r \in \mathbb{R}$ .

A linear mapping  $\ell : V \rightarrow \mathbb{R}$  is called a **linear functional** on  $V$ .

If  $V_1 = V_2$  (or if both  $V_1$  and  $V_2$  are functional spaces) then a linear mapping  $L : V_1 \rightarrow V_2$  is called a **linear operator**.

## Linear mapping = linear transformation = linear function

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*Remark.* A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = ax + b$  is a linear transformation of the vector space  $\mathbb{R}$  if and only if  $b = 0$ .

## Basic properties of linear transformations

Let  $L : V_1 \rightarrow V_2$  be a linear mapping.

- $L(r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k) = r_1L(\mathbf{v}_1) + \cdots + r_kL(\mathbf{v}_k)$   
for all  $k \geq 1$ ,  $\mathbf{v}_1, \dots, \mathbf{v}_k \in V_1$ , and  $r_1, \dots, r_k \in \mathbb{R}$ .

$$L(r_1\mathbf{v}_1 + r_2\mathbf{v}_2) = L(r_1\mathbf{v}_1) + L(r_2\mathbf{v}_2) = r_1L(\mathbf{v}_1) + r_2L(\mathbf{v}_2),$$

$$L(r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + r_3\mathbf{v}_3) = L(r_1\mathbf{v}_1 + r_2\mathbf{v}_2) + L(r_3\mathbf{v}_3) = \\ = r_1L(\mathbf{v}_1) + r_2L(\mathbf{v}_2) + r_3L(\mathbf{v}_3), \text{ and so on.}$$

- $L(\mathbf{0}_1) = \mathbf{0}_2$ , where  $\mathbf{0}_1$  and  $\mathbf{0}_2$  are zero vectors in  $V_1$  and  $V_2$ , respectively.

$$L(\mathbf{0}_1) = L(0\mathbf{0}_1) = 0L(\mathbf{0}_1) = \mathbf{0}_2.$$

- $L(-\mathbf{v}) = -L(\mathbf{v})$  for any  $\mathbf{v} \in V_1$ .

$$L(-\mathbf{v}) = L((-1)\mathbf{v}) = (-1)L(\mathbf{v}) = -L(\mathbf{v}).$$

## Examples of linear mappings

- *Scaling*  $L : V \rightarrow V$ ,  $L(\mathbf{v}) = s\mathbf{v}$ , where  $s \in \mathbb{R}$ .

$$L(\mathbf{x} + \mathbf{y}) = s(\mathbf{x} + \mathbf{y}) = s\mathbf{x} + s\mathbf{y} = L(\mathbf{x}) + L(\mathbf{y}),$$

$$L(r\mathbf{x}) = s(r\mathbf{x}) = r(s\mathbf{x}) = rL(\mathbf{x}).$$

- *Dot product with a fixed vector*

$$\ell : \mathbb{R}^n \rightarrow \mathbb{R}, \ell(\mathbf{v}) = \mathbf{v} \cdot \mathbf{v}_0, \text{ where } \mathbf{v}_0 \in \mathbb{R}^n.$$

$$\ell(\mathbf{x} + \mathbf{y}) = (\mathbf{x} + \mathbf{y}) \cdot \mathbf{v}_0 = \mathbf{x} \cdot \mathbf{v}_0 + \mathbf{y} \cdot \mathbf{v}_0 = \ell(\mathbf{x}) + \ell(\mathbf{y}),$$

$$\ell(r\mathbf{x}) = (r\mathbf{x}) \cdot \mathbf{v}_0 = r(\mathbf{x} \cdot \mathbf{v}_0) = r\ell(\mathbf{x}).$$

- *Cross product with a fixed vector*

$$L : \mathbb{R}^3 \rightarrow \mathbb{R}^3, L(\mathbf{v}) = \mathbf{v} \times \mathbf{v}_0, \text{ where } \mathbf{v}_0 \in \mathbb{R}^3.$$

- *Multiplication by a fixed matrix*

$$L : \mathbb{R}^n \rightarrow \mathbb{R}^m, L(\mathbf{v}) = A\mathbf{v}, \text{ where } A \text{ is an } m \times n \text{ matrix and all vectors are column vectors.}$$