MATH 323

Linear Algebra

Lecture 16:
Linear transformations (continued).
General linear equations.

Linear transformation

Definition. Given vector spaces V_1 and V_2, a mapping $L : V_1 \to V_2$ is **linear** if

$$\boxed{L(\mathbf{x}+\mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y}),}$$

$$\boxed{L(r\mathbf{x}) = rL(\mathbf{x})}$$

for any $\mathbf{x}, \mathbf{y} \in V_1$ and $r \in \mathbb{R}$.

Basic properties of linear mappings:

- $L(r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k) = r_1L(\mathbf{v}_1) + \cdots + r_kL(\mathbf{v}_k)$ for all $k \geq 1$, $\mathbf{v}_1, \ldots, \mathbf{v}_k \in V_1$, and $r_1, \ldots, r_k \in \mathbb{R}$.
- $L(\mathbf{0}_1) = \mathbf{0}_2$, where $\mathbf{0}_1$ and $\mathbf{0}_2$ are zero vectors in V_1 and V_2, respectively.
- $L(-\mathbf{v}) = -L(\mathbf{v})$ for any $\mathbf{v} \in V_1$.

Examples of linear mappings

- *Scaling* $L : V \to V$, $L(\mathbf{v}) = s\mathbf{v}$, where $s \in \mathbb{R}$.
- *Dot product with a fixed vector*
$\ell : \mathbb{R}^n \to \mathbb{R}$, $\ell(\mathbf{v}) = \mathbf{v} \cdot \mathbf{v}_0$, where $\mathbf{v}_0 \in \mathbb{R}^n$.
- *Cross product with a fixed vector*
$L : \mathbb{R}^3 \to \mathbb{R}^3$, $L(\mathbf{v}) = \mathbf{v} \times \mathbf{v}_0$, where $\mathbf{v}_0 \in \mathbb{R}^3$.
- *Multiplication by a fixed matrix*
$L : \mathbb{R}^n \to \mathbb{R}^m$, $L(\mathbf{v}) = A\mathbf{v}$, where A is an $m \times n$ matrix and all vectors are column vectors.
- *Coordinate mapping*
$L : V \to \mathbb{R}^n$, $L(\mathbf{v}) =$ coordinates of $\mathbf{v}$ relative to an ordered basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ for the vector space V.

Linear mappings of functional vector spaces

- *Evaluation at a fixed point*

$\ell : F(\mathbb{R}) \to \mathbb{R}$, $\ell(f) = f(a)$, where $a \in \mathbb{R}$.

- *Multiplication by a fixed function*

$L : F(\mathbb{R}) \to F(\mathbb{R})$, $L(f) = gf$, where $g \in F(\mathbb{R})$.

- *Differentiation* $D : C^1(\mathbb{R}) \to C(\mathbb{R})$, $D(f) = f'$.

$D(f+g) = (f+g)' = f' + g' = D(f) + D(g)$,
$D(rf) = (rf)' = rf' = rD(f)$.

- *Integration over a finite interval*

$\ell : C(\mathbb{R}) \to \mathbb{R}$, $\ell(f) = \int_a^b f(x)\,dx$, where $a, b \in \mathbb{R}$, $a < b$.

More properties of linear mappings

- If a linear mapping $L : V \to W$ is invertible then the inverse mapping $L^{-1} : W \to V$ is also linear.

- If $L : V \to W$ and $M : W \to X$ are linear mappings then the composition $M \circ L : V \to X$ is also linear.

- If $L_1 : V \to W$ and $L_2 : V \to W$ are linear mappings then the sum $L_1 + L_2$ is also linear.

Linear differential operators

- *Ordinary differential operator*

$$L : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}), \qquad L = g_0 \frac{d^2}{dx^2} + g_1 \frac{d}{dx} + g_2,$$

where g_0, g_1, g_2 are smooth functions on $\mathbb{R}$.
That is, $L(f) = g_0 f'' + g_1 f' + g_2 f$.

- *Laplace's operator* $\Delta : C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$,

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

(a.k.a. the Laplacian; also denoted by ∇^2).

Linear integral operators

- *Anti-derivative*

$$L : C[a,b] \to C^1[a,b], \quad (Lf)(x) = \int_a^x f(y)\,dy.$$

- *Hilbert-Schmidt operator*

$$L : C[a,b] \to C[c,d], \quad (Lf)(x) = \int_a^b K(x,y) f(y)\,dy,$$

where $K \in C([c,d] \times [a,b])$.

- *Laplace transform*

$$\mathcal{L} : BC(0,\infty) \to C(0,\infty), \quad (\mathcal{L}f)(x) = \int_0^\infty e^{-xy} f(y)\,dy.$$

Examples. $\mathcal{M}_{m,n}(\mathbb{R})$: the space of $m \times n$ matrices.

- $\alpha : \mathcal{M}_{m,n}(\mathbb{R}) \to \mathcal{M}_{n,m}(\mathbb{R})$, $\alpha(A) = A^T$.

$\alpha(A+B) = \alpha(A) + \alpha(B) \iff (A+B)^T = A^T + B^T$.
$\alpha(rA) = r\,\alpha(A) \iff (rA)^T = rA^T$.
Hence α is linear.

- $\beta : \mathcal{M}_{2,2}(\mathbb{R}) \to \mathbb{R}$, $\beta(A) = \det A$.

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Then $A + B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

We have $\det(A) = \det(B) = 0$ while $\det(A+B) = 1$.
Hence $\beta(A+B) \neq \beta(A) + \beta(B)$ so that β is not linear.

Range and kernel

Let V, W be vector spaces and $L : V \to W$ be a linear map.

Definition. The **range** (or **image**) of L is the set of all vectors $\mathbf{w} \in W$ such that $\mathbf{w} = L(\mathbf{v})$ for some $\mathbf{v} \in V$. The range of L is denoted $L(V)$.

The **kernel** of L, denoted $\ker L$, is the set of all vectors $\mathbf{v} \in V$ such that $L(\mathbf{v}) = \mathbf{0}$.

Theorem (i) If V_0 is a subspace of V then $L(V_0)$ is a subspace of W. **(ii)** If W_0 is a subspace of W then $L^{-1}(W_0)$ is a subspace of V.

Corollary (i) The range of L is a subspace of W. **(ii)** The kernel of L is a subspace of V.

Example. $L : \mathbb{R}^3 \to \mathbb{R}^3$, $L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & -1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

The kernel $\ker(L)$ is the nullspace of the matrix.

$$L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + y\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + z\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

The range $L(\mathbb{R}^3)$ is the column space of the matrix.

Example. $L : \mathbb{R}^3 \to \mathbb{R}^3$, $L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & -1 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$

The range of L is spanned by vectors $(1,1,1)$, $(0,2,0)$, and $(-1,-1,-1)$. It follows that $L(\mathbb{R}^3)$ is the plane spanned by $(1,1,1)$ and $(0,1,0)$.

To find $\ker(L)$, we apply row reduction to the matrix:

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & -1 \\ 1 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence $(x,y,z) \in \ker(L)$ if $x - z = y = 0$.
It follows that $\ker(L)$ is the line spanned by $(1,0,1)$.

More examples

- $L : \mathcal{M}_{2,2}(\mathbb{R}) \to \mathcal{M}_{2,2}(\mathbb{R})$, $L(A) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} A$.

$$L \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}.$$

The range of L is the subspace of matrices with the zero second row, $\ker L$ is the same as the range $\implies L(L(A)) = O$.

- $D : \mathcal{P}_4 \to \mathcal{P}_4$, $(Dp)(x) = p'(x)$.

$$p(x) = ax^3 + bx^2 + cx + d \implies (Dp)(x) = 3ax^2 + 2bx + c$$

The range of D is $\mathcal{P}_3$, $\ker D = \mathcal{P}_1$.

Example. $L: C^3(\mathbb{R}) \to C(\mathbb{R}),\ L(u) = u''' - 2u'' + u'.$

According to the theory of differential equations, the initial value problem

$$\begin{cases} u'''(x) - 2u''(x) + u'(x) = g(x), \quad x \in \mathbb{R}, \\ u(a) = b_0, \\ u'(a) = b_1, \\ u''(a) = b_2 \end{cases}$$

has a unique solution for any $g \in C(\mathbb{R})$ and any $b_0, b_1, b_2 \in \mathbb{R}$. It follows that $L(C^3(\mathbb{R})) = C(\mathbb{R})$.

Also, the initial data evaluation $I(u) = (u(a), u'(a), u''(a))$, which is a linear mapping $I: C^3(\mathbb{R}) \to \mathbb{R}^3$, becomes invertible when restricted to $\ker(L)$. Hence $\dim \ker(L) = 3$ since any invertible linear transformation maps a basis to a basis.

It is easy to check that $L(xe^x) = L(e^x) = L(1) = 0$. Besides, the functions xe^x, e^x, and 1 are linearly independent (use Wronskian). It follows that $\ker(L) = \mathrm{Span}(xe^x, e^x, 1)$.

General linear equation

Definition. A **linear equation** is an equation of the form

$$\boxed{L(\mathbf{x}) = \mathbf{b},}$$

where $L : V \to W$ is a linear mapping, $\mathbf{b}$ is a given vector from W, and $\mathbf{x}$ is an unknown vector from V.

The range of L is the set of all vectors $\mathbf{b} \in W$ such that the equation $L(\mathbf{x}) = \mathbf{b}$ has a solution.

The kernel of L is the solution set of the **homogeneous** linear equation $L(\mathbf{x}) = \mathbf{0}$.

Theorem If the linear equation $L(\mathbf{x}) = \mathbf{b}$ is solvable and $\dim \ker L < \infty$, then the general solution is

$$\mathbf{x}_0 + t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k,$$

where $\mathbf{x}_0$ is a particular solution, $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is a basis for the kernel of L, and $t_1, \ldots, t_k$ are arbitrary scalars.

Example. $\begin{cases} x+y+z=4, \\ x+2y=3. \end{cases}$

$$L:\mathbb{R}^3\to\mathbb{R}^2, \quad L\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1&1&1\\1&2&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}.$$

Linear equation: $L(\mathbf{x})=\mathbf{b}$, where $\mathbf{b}=\begin{pmatrix}4\\3\end{pmatrix}$.

$$\left(\begin{array}{ccc|c}1&1&1&4\\1&2&0&3\end{array}\right)\to\left(\begin{array}{ccc|c}\color{red}{1}&1&1&4\\0&\color{red}{1}&-1&-1\end{array}\right)\to\left(\begin{array}{ccc|c}\color{red}{1}&0&2&5\\0&\color{red}{1}&-1&-1\end{array}\right)$$

$$\begin{cases} x+2z=5 \\ y-z=-1 \end{cases} \iff \begin{cases} x=5-2z \\ y=-1+z \end{cases}$$

$$(x,y,z)=(5-2t,-1+t,t)=(5,-1,0)+t(-2,1,1).$$

Example. $u'''(x) - 2u''(x) + u'(x) = e^{2x}$.

Linear operator $L : C^3(\mathbb{R}) \to C(\mathbb{R})$,
$Lu = u''' - 2u'' + u'$.

Linear equation: $Lu = b$, where $b(x) = e^{2x}$.

We already know that functions xe^x, e^x and 1 form a basis for the kernel of L. It remains to find a particular solution.

$L(e^{2x}) = 8e^{2x} - 2(4e^{2x}) + 2e^{2x} = 2e^{2x}$.
Since L is a linear operator, $L\left(\frac{1}{2}e^{2x}\right) = e^{2x}$.

Particular solution: $u_0(x) = \frac{1}{2}e^{2x}$.

Thus the general solution is

$$u(x) = \frac{1}{2}e^{2x} + t_1 xe^x + t_2 e^x + t_3.$$