MATH 323

Linear Algebra

# Lecture 18a:
# Similar matrices (continued).

# Similarity of matrices

*Definition.* An $n \times n$ matrix $B$ is said to be **similar** to an $n \times n$ matrix $A$ if $\boxed{B = S^{-1}AS}$ for some nonsingular $n \times n$ matrix $S$.

*Remark.* Two $n \times n$ matrices are similar if and only if they represent the same linear operator on $\mathbb{R}^n$ with respect to different bases.

**Theorem** Similarity is an *equivalence relation*, which means that **(i)** any square matrix $A$ is similar to itself;
**(ii)** if $B$ is similar to $A$, then $A$ is similar to $B$;
**(iii)** if $A$ is similar to $B$ and $B$ is similar to $C$, then $A$ is similar to $C$.

**Corollary** The set of $n \times n$ matrices is partitioned into disjoint subsets (called *similarity classes*) such that all matrices in the same subset are similar to each other while matrices from different subsets are never similar.

**Theorem** Similarity is an *equivalence relation*, i.e.,
**(i)** any square matrix $A$ is similar to itself;
**(ii)** if $B$ is similar to $A$, then $A$ is similar to $B$;
**(iii)** if $A$ is similar to $B$ and $B$ is similar to $C$, then $A$ is similar to $C$.

*Proof:* **(i)** $A = I^{-1}AI$.
**(ii)** If $B = S^{-1}AS$ then $A = SBS^{-1} = (S^{-1})^{-1}BS^{-1}$ $= S_1^{-1}BS_1$, where $S_1 = S^{-1}$.
**(iii)** If $A = S^{-1}BS$ and $B = T^{-1}CT$ then
$A = S^{-1}(T^{-1}CT)S = (S^{-1}T^{-1})C(TS) = (TS)^{-1}C(TS)$ $= S_2^{-1}CS_2$, where $S_2 = TS$.

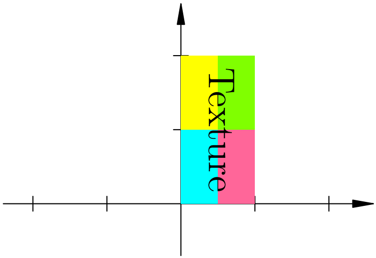
**Theorem** If $A$ and $B$ are similar matrices then they have the same **(i)** determinant, **(ii)** trace = the sum of diagonal entries, **(iii)** rank, and **(iv)** nullity.

# Linear transformations of $\mathbb{R}^2$

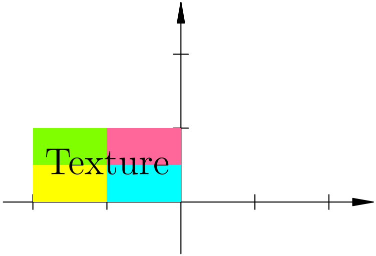
Any linear mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ is represented as multiplication of a 2-dimensional column vector by a 2×2 matrix: $f(\mathbf{x}) = A\mathbf{x}$ or

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

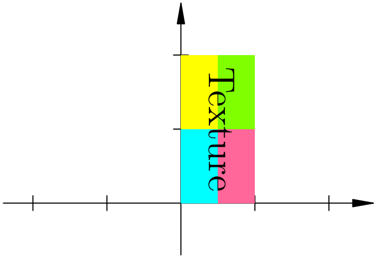
Linear transformations corresponding to particular matrices can have various geometric properties.



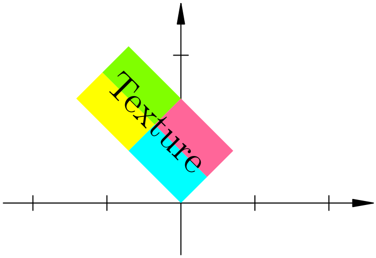

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



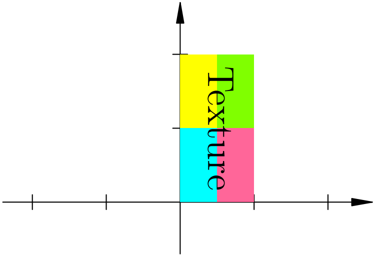

Rotation by $90^{\circ}$



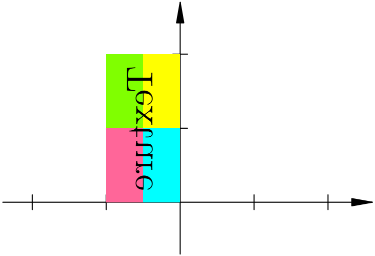

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$



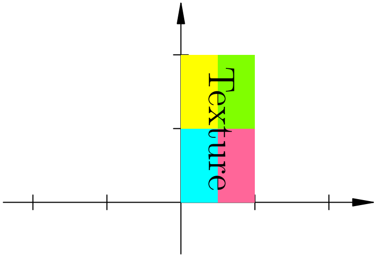

Rotation by $45^o$



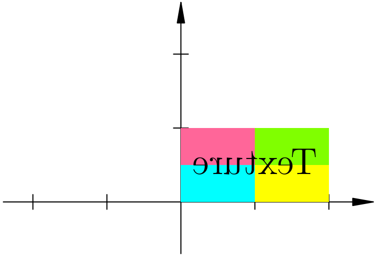

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$



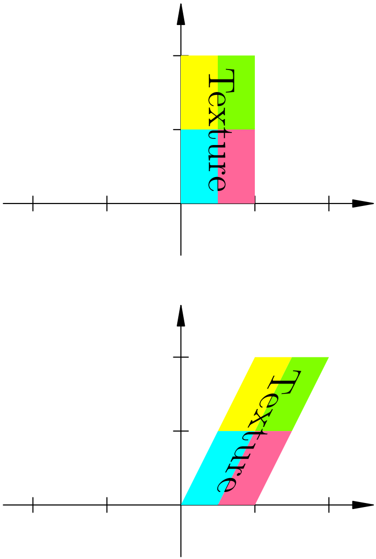

Reflection about the vertical axis




$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

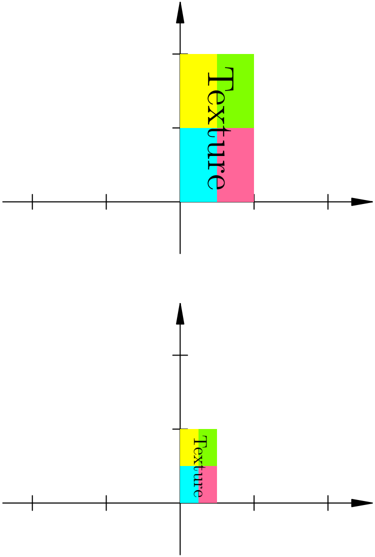



Reflection about
the line $x - y = 0$



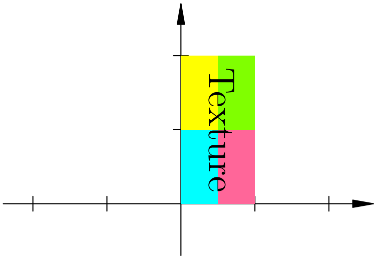

$$A = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}$$

Horizontal shear

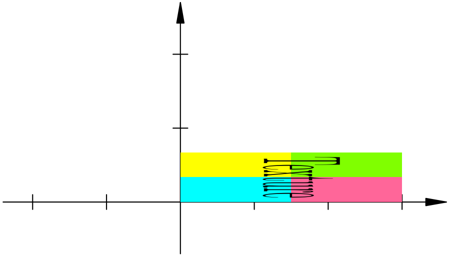



$$A = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

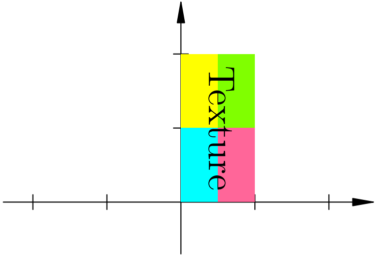
Scaling



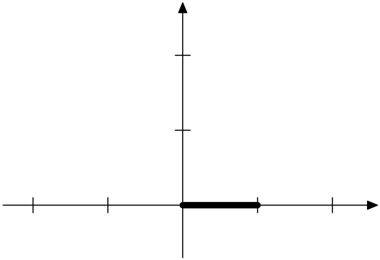

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 1/3 \end{pmatrix}$$



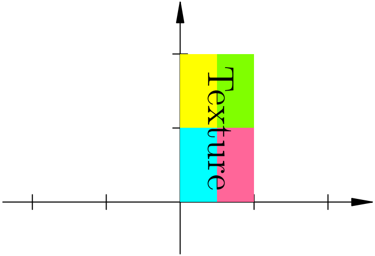
Squeeze



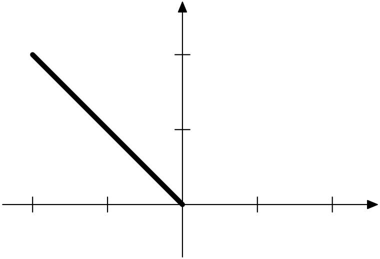

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$



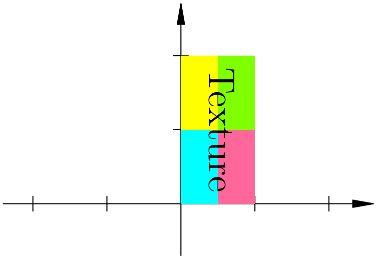
Vertical projection on
the horizontal axis



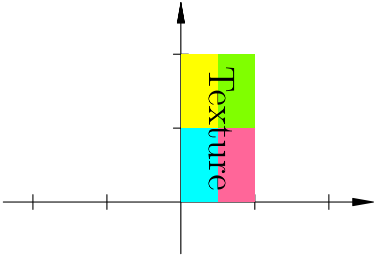

$$A = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$$



Horizontal projection on the line $x + y = 0$




$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$




Identity