

MATH 323  
Linear Algebra

**Lecture 23:**  
**Norms and inner products.**

## Norm

The notion of *norm* generalizes the notion of length of a vector in  $\mathbb{R}^n$ .

*Definition.* Let  $V$  be a vector space. A function  $\alpha : V \rightarrow \mathbb{R}$  is called a **norm** on  $V$  if it has the following properties:

- (i)  $\alpha(\mathbf{x}) \geq 0$ ,  $\alpha(\mathbf{x}) = 0$  only for  $\mathbf{x} = \mathbf{0}$  (positivity)
- (ii)  $\alpha(r\mathbf{x}) = |r| \alpha(\mathbf{x})$  for all  $r \in \mathbb{R}$  (homogeneity)
- (iii)  $\alpha(\mathbf{x} + \mathbf{y}) \leq \alpha(\mathbf{x}) + \alpha(\mathbf{y})$  (triangle inequality)

*Notation.* The norm of a vector  $\mathbf{x} \in V$  is usually denoted  $\|\mathbf{x}\|$ . Different norms on  $V$  are distinguished by subscripts, e.g.,  $\|\mathbf{x}\|_1$  and  $\|\mathbf{x}\|_2$ .

*Examples.*  $V = \mathbb{R}^n$ ,  $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ .

- $\|\mathbf{x}\|_\infty = \max(|x_1|, |x_2|, \dots, |x_n|)$ .

Positivity and homogeneity are obvious. Let  $\mathbf{x} = (x_1, \dots, x_n)$  and  $\mathbf{y} = (y_1, \dots, y_n)$ . Then  $\mathbf{x} + \mathbf{y} = (x_1 + y_1, \dots, x_n + y_n)$ .

$$|x_i + y_i| \leq |x_i| + |y_i| \leq \max_j |x_j| + \max_j |y_j|$$

$$\implies \max_j |x_j + y_j| \leq \max_j |x_j| + \max_j |y_j|$$

$$\implies \|\mathbf{x} + \mathbf{y}\|_\infty \leq \|\mathbf{x}\|_\infty + \|\mathbf{y}\|_\infty.$$

- $\|\mathbf{x}\|_1 = |x_1| + |x_2| + \dots + |x_n|$ .

Positivity and homogeneity are obvious.

The triangle inequality:  $|x_i + y_i| \leq |x_i| + |y_i|$

$$\implies \sum_j |x_j + y_j| \leq \sum_j |x_j| + \sum_j |y_j|$$

*Examples.*  $V = \mathbb{R}^n$ ,  $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ .

- $\|\mathbf{x}\|_p = (|x_1|^p + |x_2|^p + \dots + |x_n|^p)^{1/p}$ ,  $p > 0$ .

*Remark.*  $\|\mathbf{x}\|_2 =$  Euclidean length of  $\mathbf{x}$ .

**Theorem**  $\|\mathbf{x}\|_p$  is a norm on  $\mathbb{R}^n$  for any  $p \geq 1$ .

Positivity and homogeneity are still obvious (and hold for any  $p > 0$ ). The triangle inequality for  $p \geq 1$  is known as the **Minkowski inequality**:

$$\begin{aligned} (|x_1 + y_1|^p + |x_2 + y_2|^p + \dots + |x_n + y_n|^p)^{1/p} &\leq \\ &\leq (|x_1|^p + \dots + |x_n|^p)^{1/p} + (|y_1|^p + \dots + |y_n|^p)^{1/p}. \end{aligned}$$

## Normed vector space

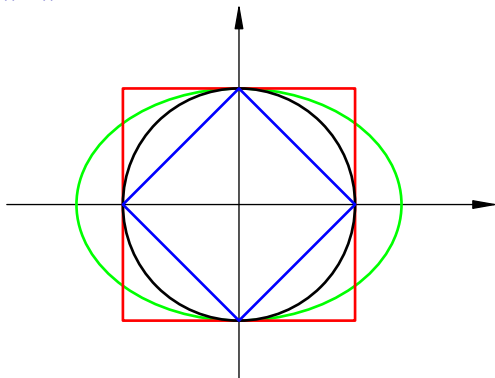
*Definition.* A **normed vector space** is a vector space endowed with a norm.

The norm defines a distance function on the normed vector space:  $\text{dist}(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$ .

Then we say that a vector  $\mathbf{x}$  is a good *approximation* of a vector  $\mathbf{x}_0$  if  $\text{dist}(\mathbf{x}, \mathbf{x}_0)$  is small.

Also, we say that a sequence  $\mathbf{x}_1, \mathbf{x}_2, \dots$  *converges* to a vector  $\mathbf{x}$  if  $\text{dist}(\mathbf{x}, \mathbf{x}_n) \rightarrow 0$  as  $n \rightarrow \infty$ .

Unit circle:  $\|\mathbf{x}\| = 1$



$$\|\mathbf{x}\| = (x_1^2 + x_2^2)^{1/2} \quad \text{black}$$

$$\|\mathbf{x}\| = \left(\frac{1}{2}x_1^2 + x_2^2\right)^{1/2} \quad \text{green}$$

$$\|\mathbf{x}\| = |x_1| + |x_2| \quad \text{blue}$$

$$\|\mathbf{x}\| = \max(|x_1|, |x_2|) \quad \text{red}$$

*Examples.*  $V = C[a, b]$ ,  $f : [a, b] \rightarrow \mathbb{R}$ .

- $\|f\|_{\infty} = \max_{a \leq x \leq b} |f(x)|.$

- $\|f\|_1 = \int_a^b |f(x)| dx.$

- $\|f\|_p = \left( \int_a^b |f(x)|^p dx \right)^{1/p}, \quad p > 0.$

**Theorem**  $\|f\|_p$  is a norm on  $C[a, b]$  for any  $p \geq 1$ .

## Inner product

The notion of *inner product* generalizes the notion of dot product of vectors in  $\mathbb{R}^n$ .

*Definition.* Let  $V$  be a vector space. A function  $\beta : V \times V \rightarrow \mathbb{R}$ , usually denoted  $\beta(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x}, \mathbf{y} \rangle$ , is called an **inner product** on  $V$  if it is positive, symmetric, and bilinear. That is, if

- (i)  $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ ,  $\langle \mathbf{x}, \mathbf{x} \rangle = 0$  only for  $\mathbf{x} = \mathbf{0}$  (positivity)
- (ii)  $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$  (symmetry)
- (iii)  $\langle r\mathbf{x}, \mathbf{y} \rangle = r\langle \mathbf{x}, \mathbf{y} \rangle$  (homogeneity)
- (iv)  $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$  (distributive law)

An **inner product space** is a vector space endowed with an inner product.



*Examples.*  $V = \mathbb{R}^n$ .

- $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n$ .

- $\langle \mathbf{x}, \mathbf{y} \rangle = d_1x_1y_1 + d_2x_2y_2 + \cdots + d_nx_ny_n$ ,  
where  $d_1, d_2, \dots, d_n > 0$ .

- $\langle \mathbf{x}, \mathbf{y} \rangle = (D\mathbf{x}) \cdot (D\mathbf{y})$ ,

where  $D$  is an invertible  $n \times n$  matrix.

*Remarks.* (a) Invertibility of  $D$  is necessary to show that  $\langle \mathbf{x}, \mathbf{x} \rangle = 0 \implies \mathbf{x} = \mathbf{0}$ .

(b) The second example is a particular case of the third one when  $D = \text{diag}(d_1^{1/2}, d_2^{1/2}, \dots, d_n^{1/2})$ .

*Problem.* Find an inner product on  $\mathbb{R}^2$  such that  $\langle \mathbf{e}_1, \mathbf{e}_1 \rangle = 2$ ,  $\langle \mathbf{e}_2, \mathbf{e}_2 \rangle = 3$ , and  $\langle \mathbf{e}_1, \mathbf{e}_2 \rangle = -1$ , where  $\mathbf{e}_1 = (1, 0)$ ,  $\mathbf{e}_2 = (0, 1)$ .

Let  $\mathbf{x} = (x_1, x_2)$ ,  $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$ .

Then  $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$ ,  $\mathbf{y} = y_1\mathbf{e}_1 + y_2\mathbf{e}_2$ .

Using bilinearity, we obtain

$$\begin{aligned}\langle \mathbf{x}, \mathbf{y} \rangle &= \langle x_1\mathbf{e}_1 + x_2\mathbf{e}_2, y_1\mathbf{e}_1 + y_2\mathbf{e}_2 \rangle \\ &= x_1\langle \mathbf{e}_1, y_1\mathbf{e}_1 + y_2\mathbf{e}_2 \rangle + x_2\langle \mathbf{e}_2, y_1\mathbf{e}_1 + y_2\mathbf{e}_2 \rangle \\ &= x_1y_1\langle \mathbf{e}_1, \mathbf{e}_1 \rangle + x_1y_2\langle \mathbf{e}_1, \mathbf{e}_2 \rangle + x_2y_1\langle \mathbf{e}_2, \mathbf{e}_1 \rangle + x_2y_2\langle \mathbf{e}_2, \mathbf{e}_2 \rangle \\ &= 2x_1y_1 - x_1y_2 - x_2y_1 + 3x_2y_2.\end{aligned}$$

It remains to check that  $\langle \mathbf{x}, \mathbf{x} \rangle > 0$  for  $\mathbf{x} \neq \mathbf{0}$ .

Indeed,  $\langle \mathbf{x}, \mathbf{x} \rangle = 2x_1^2 - 2x_1x_2 + 3x_2^2 = (x_1 - x_2)^2 + x_1^2 + 2x_2^2$ .

*Example.*  $V = \mathcal{M}_{m,n}(\mathbb{R})$ , space of  $m \times n$  matrices.

- $\langle A, B \rangle = \text{trace}(AB^T)$ .

If  $A = (a_{ij})$  and  $B = (b_{ij})$ , then  $\langle A, B \rangle = \sum_{i=1}^m \sum_{j=1}^n a_{ij}b_{ij}$ .

*Examples.*  $V = C[a, b]$ .

- $\langle f, g \rangle = \int_a^b f(x)g(x) dx$ .

- $\langle f, g \rangle = \int_a^b f(x)g(x)w(x) dx$ ,

where  $w$  is bounded, piecewise continuous, and  $w > 0$  everywhere on  $[a, b]$ .

$w$  is called the **weight** function.

**Theorem** Suppose  $\langle \mathbf{x}, \mathbf{y} \rangle$  is an inner product on a vector space  $V$ . Then

$$\langle \mathbf{x}, \mathbf{y} \rangle^2 \leq \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle \quad \text{for all } \mathbf{x}, \mathbf{y} \in V.$$

*Proof:* For any  $t \in \mathbb{R}$  let  $\mathbf{v}_t = \mathbf{x} + t\mathbf{y}$ . Then

$$\begin{aligned} \langle \mathbf{v}_t, \mathbf{v}_t \rangle &= \langle \mathbf{x} + t\mathbf{y}, \mathbf{x} + t\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{x} + t\mathbf{y} \rangle + t\langle \mathbf{y}, \mathbf{x} + t\mathbf{y} \rangle \\ &= \langle \mathbf{x}, \mathbf{x} \rangle + t\langle \mathbf{x}, \mathbf{y} \rangle + t\langle \mathbf{y}, \mathbf{x} \rangle + t^2\langle \mathbf{y}, \mathbf{y} \rangle. \end{aligned}$$

Assume that  $\mathbf{y} \neq \mathbf{0}$  and let  $t = -\frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\langle \mathbf{y}, \mathbf{y} \rangle}$ . Then

$$\langle \mathbf{v}_t, \mathbf{v}_t \rangle = \langle \mathbf{x}, \mathbf{x} \rangle + t\langle \mathbf{y}, \mathbf{x} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle - \frac{\langle \mathbf{x}, \mathbf{y} \rangle^2}{\langle \mathbf{y}, \mathbf{y} \rangle}.$$

Since  $\langle \mathbf{v}_t, \mathbf{v}_t \rangle \geq 0$ , the desired inequality follows. In the case  $\mathbf{y} = \mathbf{0}$ , we have  $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{y} \rangle = 0$ .

## Cauchy-Schwarz Inequality:

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} \sqrt{\langle \mathbf{y}, \mathbf{y} \rangle}.$$

**Corollary 1**  $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \|\mathbf{y}\|$  for all  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ .

Equivalently, for all  $x_i, y_i \in \mathbb{R}$ ,

$$(x_1y_1 + \cdots + x_ny_n)^2 \leq (x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2).$$

**Corollary 2** For any  $f, g \in C[a, b]$ ,

$$\left( \int_a^b f(x)g(x) dx \right)^2 \leq \int_a^b |f(x)|^2 dx \cdot \int_a^b |g(x)|^2 dx.$$

## Norms induced by inner products

**Theorem** Suppose  $\langle \mathbf{x}, \mathbf{y} \rangle$  is an inner product on a vector space  $V$ . Then  $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$  is a norm.

*Proof:* Positivity is obvious. Homogeneity:

$$\|r\mathbf{x}\| = \sqrt{\langle r\mathbf{x}, r\mathbf{x} \rangle} = \sqrt{r^2 \langle \mathbf{x}, \mathbf{x} \rangle} = |r| \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}.$$

Triangle inequality (follows from Cauchy-Schwarz's):

$$\begin{aligned} \|\mathbf{x} + \mathbf{y}\|^2 &= \langle \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle \\ &= \langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle \\ &\leq \langle \mathbf{x}, \mathbf{x} \rangle + |\langle \mathbf{x}, \mathbf{y} \rangle| + |\langle \mathbf{y}, \mathbf{x} \rangle| + \langle \mathbf{y}, \mathbf{y} \rangle \\ &\leq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\| \|\mathbf{y}\| + \|\mathbf{y}\|^2 = (\|\mathbf{x}\| + \|\mathbf{y}\|)^2. \end{aligned}$$

*Examples.* • The length of a vector in  $\mathbb{R}^n$ ,

$$\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2},$$

is the norm induced by the dot product

$$\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

• The norm  $\|f\|_2 = \left( \int_a^b |f(x)|^2 dx \right)^{1/2}$  on the vector space  $C[a, b]$  is induced by the inner product

$$\langle f, g \rangle = \int_a^b f(x)g(x) dx.$$

## Angle

Let  $V$  be an inner product space with an inner product  $\langle \cdot, \cdot \rangle$  and the induced norm  $\| \cdot \|$ . Then

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \| \mathbf{x} \| \| \mathbf{y} \|$$

for all  $\mathbf{x}, \mathbf{y} \in V$  (the Cauchy-Schwarz inequality).

Therefore we can define the **angle** between nonzero vectors in  $V$  by

$$\angle(\mathbf{x}, \mathbf{y}) = \arccos \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\| \mathbf{x} \| \| \mathbf{y} \|}.$$

Then  $\langle \mathbf{x}, \mathbf{y} \rangle = \| \mathbf{x} \| \| \mathbf{y} \| \cos \angle(\mathbf{x}, \mathbf{y})$ .

In particular, vectors  $\mathbf{x}$  and  $\mathbf{y}$  are **orthogonal** (denoted  $\mathbf{x} \perp \mathbf{y}$ ) if  $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ .