MATH 323

Linear Algebra

Lecture 26:
Review for the final exam.

Topics for the final exam: Part I

Elementary linear algebra (L/dP 1.1–1.5, 2.1–2.2)

- Systems of linear equations: elementary operations, Gaussian elimination, back substitution.

- Matrix of coefficients and augmented matrix. Elementary row operations, row echelon form and reduced row echelon form.

- Matrix algebra. Inverse matrix.

- Determinants: explicit formulas for 2×2 and 3×3 matrices, row and column expansions, elementary row and column operations.

Topics for the final exam: Part II

Abstract linear algebra (L/dP 3.1–3.6, 4.1–4.3)

- Vector spaces (vectors, matrices, polynomials, functional spaces).
- Subspaces. Nullspace, column space, and row space of a matrix.
- Span, spanning set. Linear independence.
- Bases and dimension.
- Rank and nullity of a matrix.
- Coordinates relative to a basis.
- Change of basis, transition matrix.
- Linear transformations.
- Matrix of a linear transformation.
- Change of basis for a linear operator.
- Similarity of matrices.

Topics for the final exam: Parts III–IV

Advanced linear algebra (L/dP 5.1–5.7, 6.1–6.4)

- Euclidean structure in $\mathbb{R}^n$ (length, angle, dot product).
- Inner products and norms.
- Orthogonal complement, orthogonal projection.
- Least squares problems.
- The Gram-Schmidt orthogonalization process.
- Orthogonal polynomials.

- Eigenvalues, eigenvectors, eigenspaces.
- Characteristic polynomial.
- Bases of eigenvectors, diagonalization.
- Complex eigenvalues and eigenvectors.
- Orthogonal matrices.
- Rigid motions, rotations in space.

Problem. Consider a system of linear equations in variables x, y, z:

$$\begin{cases} x + 2y - z = 1, \\ 2x + 3y + z = 3, \\ x + 3y + az = 0, \\ x + y + 2z = b. \end{cases}$$

Find values of the parameters a and b for which the system has infinitely many solutions, and solve the system for these values.

To determine the number of solutions for the system, we convert its augmented matrix to row echelon form using elementary row operations:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 3 & 1 & 3 \\ 1 & 3 & a & 0 \\ 1 & 1 & 2 & b \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & 3 & 1 \\ 1 & 3 & a & 0 \\ 1 & 1 & 2 & b \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 1 & a+1 & -1 \\ 1 & 1 & 2 & b \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 1 & a+1 & -1 \\ 0 & -1 & 3 & b-1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & a+1 & -1 \\ 0 & -1 & 3 & b-1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & a+4 & 0 \\ 0 & -1 & 3 & b-1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & a+4 & 0 \\ 0 & 0 & 0 & b-2 \end{array}\right).$$

Now the augmented matrix is in row echelon form (except for the case $a = -4$, $b \neq 2$ when one also needs to exchange the last two rows).

If $b \neq 2$, then there is a leading entry in the rightmost column, which indicates inconsistency.

In the case $b = 2$, the system is consistent. If, additionally, $a \neq -4$ then there is a leading entry in each of the first three columns, which implies uniqueness of the solution.

Thus the system has infinitely many solutions only if $a = -4$ and $b = 2$.

Thus the system has infinitely many solutions only if $a = -4$ and $b = 2$. To find the solutions, we proceed to reduced row echelon form (for these particular values of parameters):

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 5 & 3 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The latter matrix is the augmented matrix of the following system of linear equations (which is equivalent to the given one):

$$\begin{cases} x + 5z = 3, \\ y - 3z = -1 \end{cases} \iff \begin{cases} x = -5z + 3, \\ y = 3z - 1. \end{cases}$$

The general solution is $(x, y, z) = (-5t + 3,\ 3t - 1,\ t)$ $= (3, -1, 0) + t(-5, 3, 1), \quad t \in \mathbb{R}$.

Problem. Consider a linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ given by $L(\mathbf{v}) = (\mathbf{v} \cdot \mathbf{v}_1)\mathbf{v}_2$, where $\mathbf{v}_1 = (1, 1, 1)$, $\mathbf{v}_2 = (1, 2, 2)$.
Find the matrix of L.

Let A denote the matrix of the linear operator L. The consecutive columns of A are vectors $L(\mathbf{e}_1), L(\mathbf{e}_2), L(\mathbf{e}_3)$, where $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$ is the standard basis for $\mathbb{R}^3$.

Given $\mathbf{v} = (x, y, z) \in \mathbb{R}^3$, we have that $\mathbf{v} \cdot \mathbf{v}_1 = x + y + z$ and $L(\mathbf{v}) = \big(x + y + z,\, 2(x + y + z),\, 2(x + y + z)\big)$. It follows that $L(\mathbf{e}_1) = L(\mathbf{e}_2) = L(\mathbf{e}_3) = (1, 2, 2)$. Consequently,

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}.$$

Problem. Consider a linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ given by $L(\mathbf{v}) = (\mathbf{v} \cdot \mathbf{v}_1)\mathbf{v}_2$, where $\mathbf{v}_1 = (1, 1, 1)$, $\mathbf{v}_2 = (1, 2, 2)$.
Find the matrix of L.

Alternative solution: Given a vector $\mathbf{v} = (x, y, z) \in \mathbb{R}^3$, let $\alpha = \mathbf{v} \cdot \mathbf{v}_1$ and $(x_1, y_1, z_1) = L(\mathbf{v})$. In terms of matrix algebra, we have

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} (\alpha) = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

(note that scalar multiplication of a column vector is equivalent to multiplication by a 1×1 matrix but the matrix has to be on the right as otherwise the matrix product is not defined). It follows that the matrix of the operator L is

$$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}.$$

Problem. Consider a linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v}$, where $\mathbf{v}_0 = (3/5, 0, -4/5)$.

(a) Find the matrix A of the operator L.
(b) Find the range and kernel of L.
(c) Find the eigenvalues of L.
(d) Find the matrix of the operator L^{2022} (L applied 2022 times).

$$L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v}, \quad \mathbf{v}_0 = (3/5, 0, -4/5).$$

Let $\mathbf{v} = (x, y, z) = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$. Then

$$L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 3/5 & 0 & -4/5 \\ x & y & z \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -4/5 \\ y & z \end{vmatrix} \mathbf{e}_1 - \begin{vmatrix} 3/5 & -4/5 \\ x & z \end{vmatrix} \mathbf{e}_2 + \begin{vmatrix} 3/5 & 0 \\ x & y \end{vmatrix} \mathbf{e}_3$$

$$= \tfrac{4}{5}y\mathbf{e}_1 - \left(\tfrac{4}{5}x + \tfrac{3}{5}z\right)\mathbf{e}_2 + \tfrac{3}{5}y\mathbf{e}_3 = \left(\tfrac{4}{5}y, -\tfrac{4}{5}x - \tfrac{3}{5}z, \tfrac{3}{5}y\right).$$

In particular, $L(\mathbf{e}_1) = \left(0, -\frac{4}{5}, 0\right)$, $L(\mathbf{e}_2) = \left(\frac{4}{5}, 0, \frac{3}{5}\right)$, $L(\mathbf{e}_3) = \left(0, -\frac{3}{5}, 0\right)$.

Therefore $A = \begin{pmatrix} 0 & 4/5 & 0 \\ -4/5 & 0 & -3/5 \\ 0 & 3/5 & 0 \end{pmatrix}$.

The range of the operator L is spanned by columns of the matrix A. It follows that $\mathrm{Range}(L)$ is the plane spanned by $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = (4, 0, 3)$.

The kernel of L is the nullspace of the matrix A, i.e., the solution set for the equation $A\mathbf{x} = \mathbf{0}$.

$$\begin{pmatrix} 0 & 4/5 & 0 \\ -4/5 & 0 & -3/5 \\ 0 & 3/5 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3/4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\implies x + \tfrac{3}{4}z = y = 0 \implies \mathbf{x} = t(-3/4, 0, 1).$$

Alternatively, the kernel of L is the set of vectors $\mathbf{v} \in \mathbb{R}^3$ such that $L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v} = \mathbf{0}$.
It follows that this is the line spanned by $\mathbf{v}_0 = (3/5, 0, -4/5)$.

Characteristic polynomial of the matrix A:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 4/5 & 0 \\ -4/5 & -\lambda & -3/5 \\ 0 & 3/5 & -\lambda \end{vmatrix}$$

$$= -\lambda^3 - (3/5)^2\lambda - (4/5)^2\lambda = -\lambda^3 - \lambda = -\lambda(\lambda^2 + 1).$$

The eigenvalues are 0, i, and $-i$.

The matrix of the operator L^{2022} is A^{2022}.

Since the matrix A has eigenvalues 0, i, and $-i$, it is diagonalizable in $\mathbb{C}^3$. Namely, $A = UDU^{-1}$, where U is an invertible matrix with complex entries and

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}.$$

Then $A^{2022} = UD^{2022}U^{-1}$. We have that $D^{2022} =$ $= \mathrm{diag}\big(0, i^{2022}, (-i)^{2022}\big) = \mathrm{diag}(0, -1, -1) = D^2$.

Hence

$$A^{2022} = UD^2U^{-1} = A^2 = \begin{pmatrix} -0.64 & 0 & -0.48 \\ 0 & -1 & 0 \\ -0.48 & 0 & -0.36 \end{pmatrix}.$$

Problem. Find the distance from the point $\mathbf{y} = (0,0,0,1)$ to the subspace $V \subset \mathbb{R}^4$ spanned by vectors $\mathbf{x}_1 = (1,-1,1,-1)$, $\mathbf{x}_2 = (1,1,3,-1)$, and $\mathbf{x}_3 = (-3,7,1,3)$.

First we apply the Gram-Schmidt process to vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ and obtain an orthogonal basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ for the subspace V. Next we compute the orthogonal projection $\mathbf{p}$ of the vector $\mathbf{y}$ onto V:

$$\mathbf{p} = \frac{\langle \mathbf{y}, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 + \frac{\langle \mathbf{y}, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2 + \frac{\langle \mathbf{y}, \mathbf{v}_3 \rangle}{\langle \mathbf{v}_3, \mathbf{v}_3 \rangle}\mathbf{v}_3.$$

Then the distance from $\mathbf{y}$ to V equals $\|\mathbf{y} - \mathbf{p}\|$.

Alternatively, we can apply the Gram-Schmidt process to vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{y}$. We should obtain an orthogonal system $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$. Then the desired distance will be $\|\mathbf{v}_4\|$.

$\mathbf{x}_1 = (1, -1, 1, -1),\ \mathbf{x}_2 = (1, 1, 3, -1),$
$\mathbf{x}_3 = (-3, 7, 1, 3),\ \mathbf{y} = (0, 0, 0, 1).$

$$\mathbf{v}_1 = \mathbf{x}_1 = (1, -1, 1, -1),$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 = (1, 1, 3, -1) - \frac{4}{4}(1, -1, 1, -1)$$

$$= (0, 2, 2, 0),$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 - \frac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} \mathbf{v}_2$$

$$= (-3, 7, 1, 3) - \frac{-12}{4}(1, -1, 1, -1) - \frac{16}{8}(0, 2, 2, 0)$$

$$= (0, 0, 0, 0).$$

The Gram-Schmidt process can be used to check linear independence of vectors! It failed because the vector $\mathbf{x}_3$ is a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$. V is a plane, not a 3-dimensional subspace. To fix things, it is enough to drop $\mathbf{x}_3$, i.e., we should orthogonalize vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}$.

$$\tilde{\mathbf{v}}_3 = \mathbf{y} - \frac{\langle \mathbf{y}, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 - \frac{\langle \mathbf{y}, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2$$

$$= (0,0,0,1) - \frac{-1}{4}(1,-1,1,-1) - \frac{0}{8}(0,2,2,0)$$

$$= (1/4, -1/4, 1/4, 3/4).$$

$$\|\tilde{\mathbf{v}}_3\| = \left|\left(\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{3}{4}\right)\right| = \frac{1}{4}\,|(1,-1,1,3)| = \frac{\sqrt{12}}{4} = \frac{\sqrt{3}}{2}.$$

Bases of eigenvectors

Let A be an $n \times n$ matrix with real entries.

- A has n distinct real eigenvalues $\implies$ a basis for $\mathbb{R}^n$ formed by eigenvectors of A
- A has complex eigenvalues $\implies$ no basis for $\mathbb{R}^n$ formed by eigenvectors of A
- A has n distinct complex eigenvalues $\implies$ a basis for $\mathbb{C}^n$ formed by eigenvectors of A
- A has multiple eigenvalues $\implies$ further information is needed

- an orthonormal basis for $\mathbb{R}^n$ formed by eigenvectors of A $\iff$ A is symmetric: $A^T = A$

Problem. For each of the following 2×2 matrices determine whether it allows

(a) a basis of eigenvectors for $\mathbb{R}^2$,
(b) a basis of eigenvectors for $\mathbb{C}^2$,
(c) an orthonormal basis of eigenvectors for $\mathbb{R}^2$.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$$ (a),(b),(c): yes

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$ (a),(b),(c): no

Problem. For each of the following 2×2 matrices determine whether it allows

(a) a basis of eigenvectors for $\mathbb{R}^2$,
(b) a basis of eigenvectors for $\mathbb{C}^2$,
(c) an orthonormal basis of eigenvectors for $\mathbb{R}^2$.

$C = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$ (a),(b): yes (c): no

$D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (b): yes (a),(c): no

Problem. Let V be the vector space spanned by functions $f_1(x) = x \sin x$, $f_2(x) = x \cos x$, $f_3(x) = \sin x$, and $f_4(x) = \cos x$.
Consider the linear operator $D : V \to V$, $D = d/dx$.

(a) Find the matrix A of the operator D relative to the basis f_1, f_2, f_3, f_4.
(b) Find the eigenvalues of A.
(c) Is the matrix A diagonalizable in $\mathbb{R}^4$ (in $\mathbb{C}^4$)?

A is a 4×4 matrix whose columns are coordinates of functions $Df_i = f_i'$ relative to the basis f_1, f_2, f_3, f_4.

$$
\begin{aligned}
f_1'(x) &= (x\sin x)' = x\cos x + \sin x = f_2(x) + f_3(x),\\
f_2'(x) &= (x\cos x)' = -x\sin x + \cos x\\
&= -f_1(x) + f_4(x),\\
f_3'(x) &= (\sin x)' = \cos x = f_4(x),\\
f_4'(x) &= (\cos x)' = -\sin x = -f_3(x).
\end{aligned}
$$

Thus $A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$

Eigenvalues of A are roots of its characteristic polynomial

$$\det(A-\lambda I)=\begin{vmatrix} -\lambda & -1 & 0 & 0 \\ 1 & -\lambda & 0 & 0 \\ 1 & 0 & -\lambda & -1 \\ 0 & 1 & 1 & -\lambda \end{vmatrix}$$

Expand the determinant by the 1st row:

$$\det(A-\lambda I)=-\lambda\begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & -1 \\ 1 & 1 & -\lambda \end{vmatrix}-(-1)\begin{vmatrix} 1 & 0 & 0 \\ 1 & -\lambda & -1 \\ 0 & 1 & -\lambda \end{vmatrix}$$

$$=\lambda^2(\lambda^2+1)+(\lambda^2+1)=(\lambda^2+1)^2=(\lambda-i)^2(\lambda+i)^2.$$

The roots are i and $-i$, both of multiplicity 2.

One can show that both eigenspaces of A are one-dimensional. The eigenspace for i is spanned by $(0, 0, i, 1)$ and the eigenspace for $-i$ is spanned by $(0, 0, -i, 1)$. It follows that the matrix A is not diagonalizable in $\mathbb{C}^4$.

There is also an indirect way to show that A is not diagonalizable in $\mathbb{C}^4$. Assume the contrary. Then $A = UPU^{-1}$, where U is an invertible matrix with complex entries and

$$P = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$$

(note that P should have the same characteristic polynomial as A). This would imply that $A^2 = UP^2U^{-1}$. But $P^2 = -I$ so that $A^2 = U(-I)U^{-1} = -I$.

Let us check if $A^2 = -I$.

$$A^2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -2 & -1 & 0 \\ 2 & 0 & 0 & -1 \end{pmatrix}.$$

Since $A^2 \neq -I$, we have a contradiction. Thus the matrix A is not diagonalizable in $\mathbb{C}^4$.