

Sample problems for Test 2
(to be worked out during the review)

Problem 1 Let $A = \begin{pmatrix} 0 & -1 & 4 & 1 \\ 1 & 1 & 2 & -1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}$.

- (i) Find the rank and the nullity of the matrix A .
- (ii) Find a basis for the row space of A , then extend this basis to a basis for \mathbb{R}^4 .
- (iii) Find a basis for the nullspace of A .

Problem 2 Let A and B be two matrices such that the product AB is well defined.

- (i) Prove that $\text{rank}(AB) \leq \text{rank}(B)$.
- (ii) Prove that $\text{rank}(AB) \leq \text{rank}(A)$.

Problem 3 Complex numbers \mathbb{C} form a vector space of (real) dimension 2. Consider a function $f : \mathbb{C} \rightarrow \mathbb{C}$ given by $f(z) = (3 + 2i)z$ for all $z \in \mathbb{C}$.

- (i) Prove that f is a linear operator on the vector space \mathbb{C} .
- (ii) Find the matrix of f relative to the basis $1, i$.

Problem 4 Let V be a subspace of $\mathcal{F}(\mathbb{R})$ spanned by functions e^x and e^{-x} . Let L be a linear operator on V such that

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

is the matrix of L relative to the basis e^x, e^{-x} . Find the matrix of L relative to the basis $\cosh x = \frac{1}{2}(e^x + e^{-x})$, $\sinh x = \frac{1}{2}(e^x - e^{-x})$.

Problem 5 Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

- (i) Find all eigenvalues of the matrix A .
- (ii) For each eigenvalue of A , find an associated eigenvector.
- (iii) Is the matrix A diagonalizable? Explain.
- (iv) Find all eigenvalues of the matrix A^2 .

Problem 6 Find a linear polynomial which is the best least squares fit to the following data:

$$\begin{array}{c|c|c|c|c|c} x & -2 & -1 & 0 & 1 & 2 \\ \hline f(x) & -3 & -2 & 1 & 2 & 5 \end{array}$$

Problem 7 Let V be a subspace of \mathbb{R}^4 spanned by the vectors $\mathbf{x}_1 = (1, 1, 1, 1)$ and $\mathbf{x}_2 = (1, 0, 3, 0)$. Find the distance from the vector $\mathbf{y} = (1, 0, 0, 0)$ to the subspaces V and V^\perp .