MATH 323 Fall 2023

Homework assignment #8

Problem 1. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator. Suppose that $L(1,2) = (-2,3)$ and $L(1,-1) = (5,2)$. Find the value of $L(7,5)$.

Problem 2. Consider a map $f : \mathcal{M}_{n,n}(\mathbb{R}) \to \mathcal{M}_{n,n}(\mathbb{R})$ given by $f(A) = A + I$ for all $n \times n$ matrices A (where I is the $n \times n$ identity matrix). Determine whether f is a linear operator.

Problem 3. Determine the kernel and range of the linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ given by $L(x,y,z) = (x,x,x)$ for all $(x,y,z) \in \mathbb{R}^3$.

Problem 4. Determine the kernel and range of the linear operator $L : \mathcal{P}_3 \to \mathcal{P}_3$ given by $L(p(x)) = xp'(x)$ for all polynomials $p(x)$ of degree less than 3.

Problem 5. Consider a linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$L(x,y,z) = (y - x,\ z - y)$$

for all $(x,y,z) \in \mathbb{R}^3$. Find a matrix A such that $L(\mathbf{v}) = A\mathbf{v}$ for every $\mathbf{v} \in \mathbb{R}^3$, where $\mathbf{v}$ and $L(\mathbf{v})$ are regarded as column vectors.

Problem 6. Consider a linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$L(x,y,z) = (2z,\ y + 3x,\ 2x - z)$$

for all $(x,y,z) \in \mathbb{R}^3$. Find a matrix A such that $L(\mathbf{v}) = A\mathbf{v}$ for every $\mathbf{v} \in \mathbb{R}^3$, where $\mathbf{v}$ and $L(\mathbf{v})$ are regarded as column vectors.

Problem 7. Consider a linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$L(x,y,z) = (2x - y - z,\ 2y - x - z,\ 2z - x - y)$$

for all $(x,y,z) \in \mathbb{R}^3$. Find the matrix A of L with respect to the standard basis, and use A to find the value of $L(1,1,1)$.

Problem 8. Let V be the subspace of $C^\infty[0,1]$ spanned by the functions $f_1(x) = e^x$, $f_2(x) = xe^x$ and $f_3(x) = x^2e^x$. Let D be the differentiation operator restricted to V. Find the matrix of D with respect to the ordered basis f_1, f_2, f_3.

Problem 9. Consider a linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

for all $(x,y,z) \in \mathbb{R}^3$. Let $\mathbf{v}_1 = (1,1,1)$, $\mathbf{v}_2 = (1,2,0)$ and $\mathbf{v}_3 = (0,-2,1)$. Find the transition matrix from the ordered basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to the standard basis, and use it to determine the matrix of L with respect to $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Problem 10. Show that if A and B are similar matrices, then $\det(A) = \det(B)$.