MATH 323
Linear Algebra

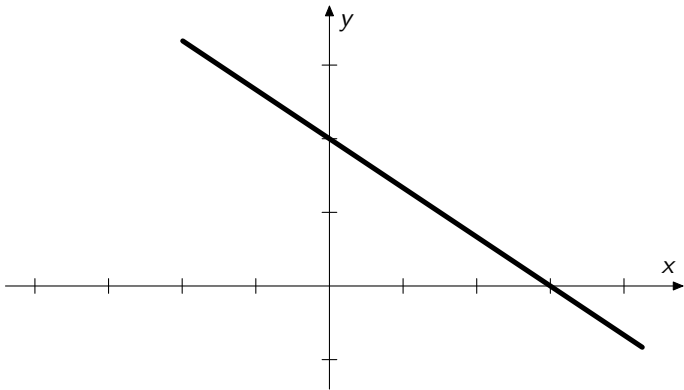**Lecture 1:**
Systems of linear equations.

Linear equation

An equation $\boxed{2x + 3y = 6}$ is called *linear* because its solution set is a straight line in $\mathbb{R}^2$.

A *solution* of the equation is a pair of numbers $(\alpha, \beta) \in \mathbb{R}^2$ such that $2\alpha + 3\beta = 6$.

For example, $(3, 0)$ and $(0, 2)$ are solutions.

Alternatively, we can write the first solution as $x = 3$, $y = 0$.

$$2x + 3y = 6$$

General equation of a line: $\boxed{ax + by = c,}$
where x, y are variables and a, b, c are constants
(except for the case $a = b = 0$).

Definition. A *linear equation* in variables
$x_1, x_2, \ldots, x_n$ is an equation of the form
$$\boxed{a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,}$$
where $a_1, \ldots, a_n,$ and b are constants.

A *solution* of the equation is an array of numbers
$(\gamma_1, \gamma_2, \ldots, \gamma_n) \in \mathbb{R}^n$ such that
$$a_1\gamma_1 + a_2\gamma_2 + \cdots + a_n\gamma_n = b.$$

System of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \qquad\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Here $x_1, x_2, \ldots, x_n$ are variables and a_{ij}, b_j are constants.

A *solution* of the system is a common solution of all equations in the system.

Plenty of problems in mathematics and real world require solving systems of linear equations.

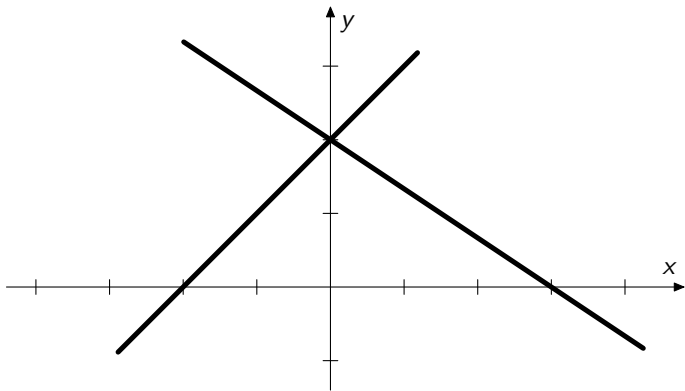**Problem** Find the point of intersection of the lines $x - y = -2$ and $2x + 3y = 6$ in $\mathbb{R}^2$.

$$\begin{cases} x - y = -2 \\ 2x + 3y = 6 \end{cases} \iff \begin{cases} x = y - 2 \\ 2x + 3y = 6 \end{cases} \iff$$

$$\begin{cases} x = y - 2 \\ 2(y - 2) + 3y = 6 \end{cases} \iff \begin{cases} x = y - 2 \\ 5y = 10 \end{cases} \iff$$

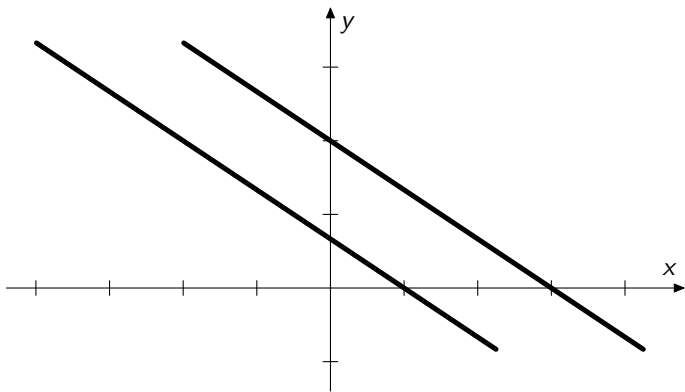$$\begin{cases} x = y - 2 \\ y = 2 \end{cases} \iff \begin{cases} x = 0 \\ y = 2 \end{cases}$$

Solution: the lines intersect at the point $(0, 2)$.

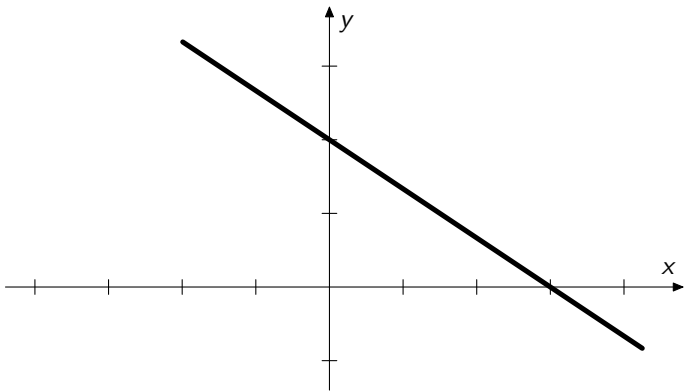*Remark.* The symbol of equivalence $\iff$ means that two systems have the same solutions.

$$\begin{cases} x - y = -2 \\ 2x + 3y = 6 \end{cases}$$

$x = 0,\ y = 2$

$$\begin{cases} 2x + 3y = 2 \\ 2x + 3y = 6 \end{cases}$$ *inconsistent system* (no solutions)

$$\begin{cases} 4x + 6y = 12 \\ 2x + 3y = 6 \end{cases} \iff 2x + 3y = 6$$

Solving systems of linear equations

Elimination method always works for systems of linear equations.

Algorithm: (1) pick a variable, solve one of the equations for it, and eliminate it from the other equations; (2) put aside the equation used in the elimination, and return to step (1).

The algorithm reduces the number of variables (as well as the number of equations), hence it stops after a finite number of steps.

After the algorithm stops, the system is simplified so that it should be clear how to complete solution.

Example.

$$\begin{cases} x - y & = 2 \\ 2x - y - z & = 3 \\ x + y + z & = 6 \end{cases}$$

Solve the 1st equation for x:

$$\begin{cases} x = y + 2 \\ 2x - y - z = 3 \\ x + y + z = 6 \end{cases}$$

Eliminate x from the 2nd and 3rd equations:

$$\begin{cases} x = y + 2 \\ 2(y + 2) - y - z = 3 \\ (y + 2) + y + z = 6 \end{cases}$$

Simplify:

$$\begin{cases} x = y + 2 \\ y - z = -1 \\ 2y + z = 4 \end{cases}$$

Now the 2nd and 3rd equations form the system of two linear equations in two variables.

Solve the 2nd equation for y:

$$\begin{cases} x = y + 2 \\ y = z - 1 \\ 2y + z = 4 \end{cases}$$

Eliminate y from the 3rd equation:

$$\begin{cases} x = y + 2 \\ y = z - 1 \\ 2(z - 1) + z = 4 \end{cases}$$

Simplify:

$$\begin{cases} x = y + 2 \\ y = z - 1 \\ 3z = 6 \end{cases}$$

The elimination is completed. Now the system is easily solved by back substitution.

That is, we find z from the 3rd equation, then substitute it in the 2nd equation and find y, then substitute y and z in the 1st equation and find x.

$$\begin{cases} x = y + 2 \\ y = z - 1 \\ z = 2 \end{cases} \qquad \begin{cases} x = y + 2 \\ y = 1 \\ z = 2 \end{cases} \qquad \begin{cases} x = 3 \\ y = 1 \\ z = 2 \end{cases}$$

System of linear equations:

$$\begin{cases} x - y & = 2 \\ 2x - y - z & = 3 \\ x + y + z & = 6 \end{cases}$$

Solution: $(x, y, z) = (3, 1, 2)$

Another example.

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 14 \end{cases}$$

Solve the 1st equation for x:

$$\begin{cases} x = -y + 2z + 1 \\ y - z = 3 \\ -x + 4y - 3z = 14 \end{cases}$$

Eliminate x from the 3rd equation:

$$\begin{cases} x = -y + 2z + 1 \\ y - z = 3 \\ -(-y + 2z + 1) + 4y - 3z = 14 \end{cases}$$

Simplify:

$$\begin{cases} x = -y + 2z + 1 \\ y - z = 3 \\ 5y - 5z = 15 \end{cases}$$

Solve the 2nd equation for y:

$$\begin{cases} x = -y + 2z + 1 \\ y = z + 3 \\ 5y - 5z = 15 \end{cases}$$

Eliminate y from the 3rd equation:

$$\begin{cases} x = -y + 2z + 1 \\ y = z + 3 \\ 5(z + 3) - 5z = 15 \end{cases}$$

Simplify:

$$\begin{cases} x = -y + 2z + 1 \\ y = z + 3 \\ 0 = 0 \end{cases}$$

The elimination is completed.

The last equation is actually $0z = 0$. Hence z is a *free variable*, i.e., it can be assigned an arbitrary value. Then y and x are found by back substitution.

$z = t$, a parameter;
$y = z + 3 = t + 3$;
$x = -y + 2z + 1 = -(t + 3) + 2t + 1 = t - 2$.

System of linear equations:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 14 \end{cases}$$

General solution:

$$(x, y, z) = (t - 2, t + 3, t), \ t \in \mathbb{R}.$$

In vector form, $(x, y, z) = (-2, 3, 0) + t(1, 1, 1)$.

The set of all solutions is a straight line in $\mathbb{R}^3$ passing through the point $(-2, 3, 0)$ in the direction $(1, 1, 1)$.