MATH 323

Linear Algebra

Lecture 11:
Linear independence (continued).
Basis and dimension.

Linear independence

Definition. Let V be a vector space. Vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V$ are called **linearly dependent** if they satisfy a relation

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{0},$$

where the coefficients $r_1, \dots, r_k \in \mathbb{R}$ are not all equal to zero. Otherwise vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are called **linearly independent**. That is, if

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{0} \implies r_1 = \cdots = r_k = 0.$$

A set $S \subset V$ is **linearly dependent** if one can find some distinct linearly dependent vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ in S. Otherwise S is **linearly independent**.

Theorem The following conditions are equivalent:
(i) vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ $(k \geq 2)$ are linearly dependent;
(ii) one of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ is a linear combination of the other $k-1$ vectors.

Proof: **(i)** $\Longrightarrow$ **(ii)** Suppose that

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{0},$$

where $r_i \neq 0$ for some i, $1 \leq i \leq k$. Then

$$\mathbf{v}_i = -\frac{r_1}{r_i}\mathbf{v}_1 - \cdots - \frac{r_{i-1}}{r_i}\mathbf{v}_{i-1} - \frac{r_{i+1}}{r_i}\mathbf{v}_{i+1} - \cdots - \frac{r_k}{r_i}\mathbf{v}_k.$$

(ii) $\Longrightarrow$ **(i)** Suppose that

$$\mathbf{v}_i = s_1\mathbf{v}_1 + \cdots + s_{i-1}\mathbf{v}_{i-1} + s_{i+1}\mathbf{v}_{i+1} + \cdots + s_k\mathbf{v}_k$$

for some scalars s_j. Then

$$s_1\mathbf{v}_1 + \cdots + s_{i-1}\mathbf{v}_{i-1} - \mathbf{v}_i + s_{i+1}\mathbf{v}_{i+1} + \cdots + s_k\mathbf{v}_k = \mathbf{0}.$$

Problem. Let $A = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$. Determine whether matrices A, A^2, and A^3 are linearly independent.

We have $A = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$, $A^2 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, $A^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The task is to check if there exist $r_1, r_2, r_3 \in \mathbb{R}$ not all zero such that $r_1 A + r_2 A^2 + r_3 A^3 = O$.

This matrix equation is equivalent to a system

$$\begin{cases} -r_1 + 0r_2 + r_3 = 0 \\ r_1 - r_2 + 0r_3 = 0 \\ -r_1 + r_2 + 0r_3 = 0 \\ 0r_1 - r_2 + r_3 = 0 \end{cases} \qquad \left(\begin{array}{rrr|r} -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The row echelon form of the augmented matrix shows there is a free variable. Hence the system has a nonzero solution so that the matrices are linearly dependent (one relation is $A + A^2 + A^3 = O$).

Problem. Show that functions e^x, e^{2x}, and e^{3x} are linearly independent in $C^\infty(\mathbb{R})$.

Suppose that $ae^x + be^{2x} + ce^{3x} = 0$ for all $x \in \mathbb{R}$, where a, b, c are constants. We have to show that $a = b = c = 0$.

Differentiate this identity twice:

$$\begin{aligned} ae^x + be^{2x} + ce^{3x} &= 0, \\ ae^x + 2be^{2x} + 3ce^{3x} &= 0, \\ ae^x + 4be^{2x} + 9ce^{3x} &= 0. \end{aligned}$$

It follows that $A(x)\mathbf{v} = \mathbf{0}$, where

$$A(x) = \begin{pmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

$$A(x) = \begin{pmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

$$\det A(x) = e^x \begin{vmatrix} 1 & e^{2x} & e^{3x} \\ 1 & 2e^{2x} & 3e^{3x} \\ 1 & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^x e^{2x} \begin{vmatrix} 1 & 1 & e^{3x} \\ 1 & 2 & 3e^{3x} \\ 1 & 4 & 9e^{3x} \end{vmatrix}$$

$$= e^x e^{2x} e^{3x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 4 & 9 \end{vmatrix}$$

$$= e^{6x} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 8 \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & 2 \\ 3 & 8 \end{vmatrix} = 2e^{6x} \neq 0.$$

Since the matrix $A(x)$ is invertible, we obtain

$$A(x)\mathbf{v} = \mathbf{0} \implies \mathbf{v} = \mathbf{0} \implies a = b = c = 0$$

Wronskian

Let $f_1, f_2, \ldots, f_n$ be smooth functions on an interval $[a, b]$. The **Wronskian** $W[f_1, f_2, \ldots, f_n]$ is a function on $[a, b]$ defined by

$$W[f_1, f_2, \ldots, f_n](x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}.$$

Theorem If $W[f_1, f_2, \ldots, f_n](x_0) \neq 0$ for some $x_0 \in [a, b]$ then the functions $f_1, f_2, \ldots, f_n$ are linearly independent in $C[a, b]$.

Basis

Definition. Let V be a vector space. Any linearly independent spanning set for V is called a **basis**.

Suppose that a set $S \subset V$ is a basis for V.
"Spanning set" means that any vector $\mathbf{v} \in V$ can be represented as a linear combination

$$\mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k,$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are distinct vectors from S and $r_1, \ldots, r_k \in \mathbb{R}$. "Linearly independent" implies that the above representation is unique:

$$\mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = r_1'\mathbf{v}_1 + r_2'\mathbf{v}_2 + \cdots + r_k'\mathbf{v}_k$$
$$\implies (r_1 - r_1')\mathbf{v}_1 + (r_2 - r_2')\mathbf{v}_2 + \cdots + (r_k - r_k')\mathbf{v}_k = \mathbf{0}$$
$$\implies r_1 - r_1' = r_2 - r_2' = \ldots = r_k - r_k' = 0$$

Basis

Definition. Let V be a vector space. Any linearly independent spanning set for V is called a **basis**.

Theorem A nonempty set $S \subset V$ is a basis for V if and only if any vector $\mathbf{v} \in V$ is *uniquely represented* as a linear combination $\mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k$, where $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are distinct vectors from S and $r_1, \ldots, r_k \in \mathbb{R}$.

Remark on uniqueness. Expansions $\mathbf{v} = 2\mathbf{v}_1 - \mathbf{v}_2$, $\mathbf{v} = -\mathbf{v}_2 + 2\mathbf{v}_1$, and $\mathbf{v} = 2\mathbf{v}_1 - \mathbf{v}_2 + 0\mathbf{v}_3$ are considered the same.

Examples. • Standard basis for $\mathbb{R}^n$:
$\mathbf{e}_1 = (1, 0, 0, \dots, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0, \dots, 0, 0)$, ...,
$\mathbf{e}_n = (0, 0, 0, \dots, 0, 1)$.
Indeed, $(x_1, x_2, \dots, x_n) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$.

• Matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
form a basis for $\mathcal{M}_{2,2}(\mathbb{R})$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

• Polynomials $1, x, x^2, \dots, x^{n-1}$ form a basis for
$\mathcal{P}_n = \{a_0 + a_1x + \cdots + a_{n-1}x^{n-1} : a_i \in \mathbb{R}\}$.

• The infinite set $\{1, x, x^2, \dots, x^n, \dots\}$ is a basis for $\mathcal{P}$, the space of all polynomials.

Let $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n$ and $r_1, r_2, \dots, r_k \in \mathbb{R}$. The vector equation $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{v}$ is equivalent to the matrix equation $A\mathbf{x} = \mathbf{v}$, where

$$A = (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k), \qquad \mathbf{x} = \begin{pmatrix} r_1 \\ \vdots \\ r_k \end{pmatrix}.$$

$$r_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} + r_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix} + \cdots + r_k \begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{nk} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \iff$$

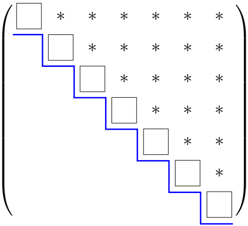
$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nk} \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_k \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \iff A\mathbf{x} = \mathbf{v}$$

Let $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n$ and $r_1, r_2, \dots, r_k \in \mathbb{R}$. The vector equation $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{v}$ is equivalent to the matrix equation $A\mathbf{x} = \mathbf{v}$, where

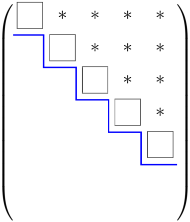
$$A = (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k), \qquad \mathbf{x} = \begin{pmatrix} r_1 \\ \vdots \\ r_k \end{pmatrix}.$$

That is, A is the $n \times k$ matrix such that vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are consecutive columns of A.

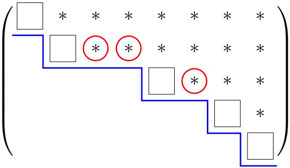
- *Vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ span $\mathbb{R}^n$ if the row echelon form of A has no zero rows.*
- *Vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ are linearly independent if the row echelon form of A has a leading entry in each column (no free variables).*



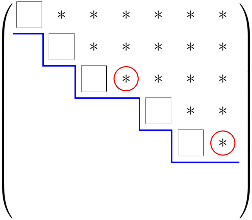
spanning
linear independence



no spanning
linear independence



spanning
no linear independence



no spanning
no linear independence

Bases for $\mathbb{R}^n$

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$.

Theorem 1 If $k < n$ then the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ do not span $\mathbb{R}^n$.

Theorem 2 If $k > n$ then the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are linearly dependent.

Theorem 3 If $k = n$ then the following conditions are equivalent:
(i) $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$;
(ii) $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a spanning set for $\mathbb{R}^n$;
(iii) $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a linearly independent set.

Example. Consider vectors $\mathbf{v}_1 = (1, -1, 1)$, $\mathbf{v}_2 = (1, 0, 0)$, $\mathbf{v}_3 = (1, 1, 1)$, and $\mathbf{v}_4 = (1, 2, 4)$ in $\mathbb{R}^3$.

Vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent (as they are not parallel), but they do not span $\mathbb{R}^3$.

Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent since

$$\begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = -\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -(-2) = 2 \neq 0.$$

Therefore $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$.

Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ span $\mathbb{R}^3$ (because $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ already span $\mathbb{R}^3$), but they are linearly dependent.

Dimension

Theorem 1 Any vector space has a basis.

Theorem 2 If a vector space V has a finite basis, then all bases for V are finite and have the same number of elements.

Definition. The **dimension** of a vector space V, denoted $\dim V$, is the number of elements in any of its bases.

Examples. • $\dim \mathbb{R}^n = n$

• $\mathcal{M}_{2,2}(\mathbb{R})$: the space of 2×2 matrices
$\dim \mathcal{M}_{2,2}(\mathbb{R}) = 4$

• $\mathcal{M}_{m,n}(\mathbb{R})$: the space of $m\times n$ matrices
$\dim \mathcal{M}_{m,n}(\mathbb{R}) = mn$

• $\mathcal{P}_n$: polynomials of degree less than n
$\dim \mathcal{P}_n = n$

• $\mathcal{P}$: the space of all polynomials
$\dim \mathcal{P} = \infty$

• $\{\mathbf{0}\}$: the trivial vector space
$\dim \{\mathbf{0}\} = 0$