MATH 409–501/503 Fall 2013

Sample problems for the final exam

Any problem may be altered or replaced by a different one!

Problem 1 (20 pts.) Suppose $E_1, E_2, E_3, \dots$ are countable sets. Prove that their union $E_1 \cup E_2 \cup E_3 \cup \dots$ is also a countable set.

Problem 2 (20 pts.) Find the following limits:

(i) $\displaystyle\lim_{x\to 0} \log \frac{1}{1+\cot(x^2)}$, (ii) $\displaystyle\lim_{x\to 64} \frac{\sqrt{x}-8}{\sqrt[3]{x}-4}$, (iii) $\displaystyle\lim_{n\to\infty} \Big(1+\frac{c}{n}\Big)^n$, where $c \in \mathbb{R}$.

Problem 3 (20 pts.) Prove that the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

converges to $\sin x$ for any $x \in \mathbb{R}$.

Problem 4 (20 pts.) Find an indefinite integral and evaluate definite integrals:

(i) $\displaystyle\int \frac{\sqrt{1+\sqrt[4]{x}}}{2\sqrt{x}}\,dx$, (ii) $\displaystyle\int_0^{\sqrt{3}} \frac{x^2+6}{x^2+9}\,dx$, (iii) $\displaystyle\int_0^{\infty} x^2 e^{-x}\,dx$.

Problem 5 (20 pts.) For each of the following series, determine whether the series converges and whether it converges absolutely:

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$, (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+2^n \cos n}{n!}$, (iii) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n \log n}$.

Bonus Problem 6 (15 pts.) Prove that an infinite product

$$\prod_{n=1}^{\infty} \frac{n^2+1}{n^2} = \frac{2}{1} \cdot \frac{5}{4} \cdot \frac{10}{9} \cdot \frac{17}{16} \cdot \dots$$

converges, that is, partial products $\prod_{k=1}^{n} \frac{k^2+1}{k^2}$ converge to a finite limit as $n \to \infty$.