MATH 409–501/503 Fall 2013

Sample problems for the final exam: Solutions

Any problem may be altered or replaced by a different one!

Problem 1 (20 pts.) Suppose $E_1, E_2, E_3, \dots$ are countable sets. Prove that their union $E_1 \cup E_2 \cup E_3 \cup \dots$ is also a countable set.

First we are going to show that the set $\mathbb{N} \times \mathbb{N}$ is countable. Consider a relation $\prec$ on the set $\mathbb{N} \times \mathbb{N}$ such that $(n_1, n_2) \prec (m_1, m_2)$ if and only if either $n_1 + n_2 < m_1 + m_2$ or else $n_1 + n_2 = m_1 + m_2$ and $n_1 < m_1$. It is easy to see that $\prec$ is a strict linear order. Moreover, for any pair $(m_1, m_2) \in \mathbb{N} \times \mathbb{N}$ there are only finitely many pairs (n_1, n_2) such that $(n_1, n_2) \prec (m_1, m_2)$. It follows that $\prec$ is a well-ordering. Now we define inductively a mapping $F : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ such that for any $n \in \mathbb{N}$ the pair $F(n)$ is the least (relative to $\prec$) pair different from $F(k)$ for all natural numbers $k < n$. It follows from the construction that F is bijective. Thus $\mathbb{N} \times \mathbb{N}$ is a countable set. By the way, the inverse mapping F^{-1} can be given explicitly by

$$F^{-1}(n_1, n_2) = \frac{(n_1 + n_2 - 2)(n_1 + n_2 - 1)}{2} + n_1, \quad n_1, n_2 \in \mathbb{N}.$$

Now suppose that $E_1, E_2, \dots$ are countable sets. Then for any $n \in \mathbb{N}$ there exists a bijective mapping $f_n : \mathbb{N} \to E_n$. Let us define a map $g : \mathbb{N} \times \mathbb{N} \to E_1 \cup E_2 \cup \dots$ by $g(n_1, n_2) = f_{n_1}(n_2)$. Obviously, g is onto. Since the set $\mathbb{N} \times \mathbb{N}$ is countable, there exists a sequence $p_1, p_2, p_3, \dots$ that is a complete list of its elements. Then the sequence $g(p_1), g(p_2), g(p_3), \dots$ contains all elements of the union $E_1 \cup E_2 \cup E_3 \cup \dots$ Although the latter sequence may include repetitions, we can choose a subsequence $\{g(p_{n_k})\}$ in which every element of the union appears exactly once. Note that the subsequence is infinite since each of the sets $E_1, E_2, \dots$ is infinite. Then the map h of $\mathbb{N}$ defined by $h(k) = g(p_{n_k})$, $k = 1, 2, \dots$, is a bijection onto $E_1 \cup E_2 \cup E_3 \cup \dots$

Problem 2 (20 pts.) Find the following limits:

(i) $\displaystyle\lim_{x\to 0} \log \frac{1}{1 + \cot(x^2)}$, (ii) $\displaystyle\lim_{x\to 64} \frac{\sqrt{x} - 8}{\sqrt[3]{x} - 4}$, (iii) $\displaystyle\lim_{n\to\infty} \left(1 + \frac{c}{n}\right)^n$, where $c \in \mathbb{R}$.

The function

$$f(x) = \log \frac{1}{1 + \cot(x^2)}$$

can be represented as the composition of 4 functions: $f_1(x) = x^2$, $f_2(y) = \cot y$, $f_3(z) = (1 + z)^{-1}$, and $f_4(u) = \log u$. Since the function f_1 is continuous, we have $\lim_{x\to 0} f_1(x) = f_1(0) = 0$. Moreover, $f_1(x) > 0$ for $x \neq 0$. Since $\lim_{y\to 0+} \cot y = +\infty$, it follows that $f_2(f_1(x)) \to +\infty$ as $x \to 0$. Further, $f_3(z) \to 0+$ as $z \to +\infty$ and $f_4(u) \to -\infty$ as $u \to 0+$. Finally, $f(x) = f_4(f_3(f_2(f_1(x)))) \to -\infty$ as $x \to 0$.

To find the second limit, consider a function $u(x) = x^{1/6}$ defined on $(0, \infty)$. Since this function is continuous at 64 and $u(64) = 2$, we obtain

$$\lim_{x \to 64} \frac{\sqrt{x} - 8}{\sqrt[3]{x} - 4} = \lim_{x \to 64} \frac{(u(x))^3 - 8}{(u(x))^2 - 4} = \lim_{y \to 2} \frac{y^3 - 8}{y^2 - 4} = \lim_{y \to 2} \frac{(y-2)(y^2 + 2y + 4)}{(y-2)(y+2)}$$

$$= \lim_{y \to 2} \frac{y^2 + 2y + 4}{y + 2} = \left. \frac{y^2 + 2y + 4}{y + 2} \right|_{y=2} = 3.$$

Given $c \in \mathbb{R}$, let $a_n = (1 + c/n)^n$ for all $n \in \mathbb{N}$. For n large enough, we have $1 + c/n > 0$ so that $a_n > 0$. Then

$$\log a_n = \log\Big(1 + \frac{c}{n}\Big)^n = n \log\Big(1 + \frac{c}{n}\Big) = \left. \frac{\log(1 + cx)}{x} \right|_{x=1/n}.$$

Since $1/n \to 0$ as $n \to \infty$ and

$$\lim_{x \to 0} \frac{\log(1 + cx)}{x} = \big(\log(1 + cx)\big)'\Big|_{x=0} = \left. \frac{c}{1 + cx} \right|_{x=0} = c,$$

we obtain that $\log a_n \to c$ as $n \to \infty$. Therefore $a_n = e^{\log a_n} \to e^c$ as $n \to \infty$.

Problem 3 (20 pts.) Prove that the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

converges to $\sin x$ for any $x \in \mathbb{R}$.

The function $f(x) = \sin x$ is infinitely differentiable on the entire real line. According to Taylor's formula, for any $x, x_0 \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_n(x, x_0),$$

where

$$R_n(x, x_0) = \frac{f^{(n+1)}(\theta)}{(n+1)!}(x - x_0)^{n+1}$$

for some $\theta = \theta(x, x_0)$ between x and x_0. Since $f'(x) = \cos x$ and $f''(x) = -\sin x = -f(x)$ for all $x \in \mathbb{R}$, it follows that $|f^{(n+1)}(\theta)| \leq 1$ for all $n \in \mathbb{N}$ and $\theta \in \mathbb{R}$. Hence $|R_n(x, x_0)| \leq |x - x_0|^{n+1}/(n+1)!$. Let us fix x and x_0. Then there exists $N \in \mathbb{N}$ such that $N \geq 2|x - x_0|$. For any natural number $n \geq N$ we have

$$|R_n(x, x_0)| \leq \frac{|x - x_0|^{n+1}}{(n+1)!} \leq \frac{|x - x_0|^N}{N!} \cdot \frac{1}{2^{n+1-N}},$$

which implies that $R_n(x, x_0) \to 0$ as $n \to \infty$. In other words, the series

$$f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \dots$$

converges to $f(x) = \sin x$ for all $x, x_0 \in \mathbb{R}$. In the case $x_0 = 0$, the sequence $\{f^{(n)}(x_0)\}$ is a periodic sequence $1, 0, -1, 0, 1, 0, -1, 0, \dots$ Consequently, this series coincides with the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

up to zero terms.

Problem 4 (20 pts.) Find an indefinite integral and evaluate definite integrals:

(i) $\displaystyle\int \frac{\sqrt{1+\sqrt[4]{x}}}{2\sqrt{x}}\,dx$, (ii) $\displaystyle\int_0^{\sqrt{3}} \frac{x^2+6}{x^2+9}\,dx$, (iii) $\displaystyle\int_0^{\infty} x^2 e^{-x}\,dx$.

To find the indefinite integral, we change the variable twice. First

$$\int \frac{\sqrt{1+\sqrt[4]{x}}}{2\sqrt{x}}\,dx = \int \sqrt{1+\sqrt[4]{x}}\,(\sqrt{x})'\,dx = \int \sqrt{1+\sqrt[4]{x}}\,d(\sqrt{x}) = \int \sqrt{1+\sqrt{u}}\,du,$$

where $u = \sqrt{x}$. Secondly, we introduce another variable $w = \sqrt{1+\sqrt{u}}$. Then $u = (w^2-1)^2$ so that $du = \big((w^2-1)^2\big)'\,dw = 2(w^2-1)\cdot 2w\,dw = (4w^3 - 4w)\,dw$. Consequently,

$$\int \frac{\sqrt{1+\sqrt[4]{x}}}{2\sqrt{x}}\,dx = \int \sqrt{1+\sqrt{u}}\,du = \int w\,du = \int (4w^4 - 4w^2)\,dw$$
$$= \frac{4}{5}w^5 - \frac{4}{3}w^3 + C = \frac{4}{5}\big(1+x^{1/4}\big)^{5/2} - \frac{4}{3}\big(1+x^{1/4}\big)^{3/2} + C.$$

To evaluate the first definite integral, we use linearity of the integral, a substitution $x = 3u$, and the fact that $(\arctan x)' = 1/(1+x^2)$:

$$\int_0^{\sqrt{3}} \frac{x^2+6}{x^2+9}\,dx = \int_0^{\sqrt{3}} \left(1 - \frac{3}{x^2+9}\right)dx = \int_0^{\sqrt{3}} 1\,dx - \int_0^{\sqrt{3}} \frac{3}{x^2+9}\,dx$$
$$= \sqrt{3} - \int_0^{\sqrt{3}/3} \frac{3}{(3u)^2+9}\,d(3u) = \sqrt{3} - \int_0^{1/\sqrt{3}} \frac{1}{u^2+1}\,du = \sqrt{3} - \arctan u\Big|_{u=0}^{1/\sqrt{3}} = \sqrt{3} - \frac{\pi}{6}.$$

To evaluate the improper integral, we integrate by parts twice:

$$\int_0^{\infty} x^2 e^{-x}\,dx = -\int_0^{\infty} x^2 (e^{-x})'\,dx = -\int_0^{\infty} x^2\,d(e^{-x}) = -x^2e^{-x}\Big|_0^{\infty} + \int_0^{\infty} e^{-x}\,d(x^2)$$
$$= \int_0^{\infty} e^{-x}(x^2)'\,dx = \int_0^{\infty} 2xe^{-x}\,dx = -\int_0^{\infty} 2x(e^{-x})'\,dx = -\int_0^{\infty} 2x\,d(e^{-x})$$
$$= -2xe^{-x}\Big|_0^{\infty} + \int_0^{\infty} e^{-x}\,d(2x) = \int_0^{\infty} 2e^{-x}\,dx = -2e^{-x}\Big|_0^{\infty} = 2.$$

Problem 5 (20 pts.) For each of the following series, determine whether the series converges and whether it converges absolutely:

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$, (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+2^n \cos n}{n!}$, (iii) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n \log n}$.

The first series diverges. Indeed,

$$\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{\big(\sqrt{n+1}-\sqrt{n}\big)\big(\sqrt{n+1}+\sqrt{n}\big)}{\big(\sqrt{n+1}+\sqrt{n}\big)^2} = \frac{1}{\big(\sqrt{n+1}+\sqrt{n}\big)^2} > \frac{1}{\big(2\sqrt{n+1}\big)^2} = \frac{1}{4(n+1)}.$$

Since the series $\sum_{n=1}^{\infty}\big(4(n+1)\big)^{-1}$ diverges, it remains to apply the Comparison Test.

Let a_n denote the n-th term of the second series. We have $a_n = b_n + c_n \cos n$, where $b_n = \sqrt{n}/n!$ and $c_n = 2^n/n!$ for all $n \in \mathbb{N}$. The series $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$ both converge, which can be verified with the Ratio Test:

$$\frac{b_{n+1}}{b_n} = \frac{\sqrt{n+1}}{(n+1)!}\left(\frac{\sqrt{n}}{n!}\right)^{-1} = \left(1+\frac{1}{n}\right)^{1/2} \cdot \frac{1}{n+1} \to 0 \quad \text{as} \quad n \to \infty,$$

$$\frac{c_{n+1}}{c_n} = \frac{2^{n+1}}{(n+1)!}\left(\frac{2^n}{n!}\right)^{-1} = \frac{2}{n+1} \to 0 \quad \text{as} \quad n \to \infty.$$

Then the series $\sum_{n=1}^{\infty}(b_n + c_n)$ converges as well. Since $|a_n| \le b_n + c_n$ for all $n \in \mathbb{N}$, the series $\sum_{n=1}^{\infty} a_n$ converges absolutely due to the Comparison Test.

The function $f(x) = (x \log x)^{-1}$ is positive and decreasing on $[2, \infty)$. Moreover, $\lim_{x\to\infty} f(x) = 0$. By the Alternating Series Test, the series $\sum_{n=2}^{\infty}(-1)^n/(n \log n)$ converges. However the convergence is not absolute due to the Integral Test:

$$\int_2^c \frac{1}{x \log x}\,dx = \int_2^c \frac{(\log x)'}{\log x}\,dx = \int_{\log 2}^{\log c} \frac{du}{u} = \log(\log c) - \log(\log 2) \to +\infty \quad \text{as} \quad c \to +\infty.$$

Bonus Problem 6 (15 pts.) Prove that an infinite product

$$\prod_{n=1}^{\infty} \frac{n^2+1}{n^2} = \frac{2}{1} \cdot \frac{5}{4} \cdot \frac{10}{9} \cdot \frac{17}{16} \cdot \dots$$

converges, that is, partial products $\prod_{k=1}^{n} \dfrac{k^2+1}{k^2}$ converge to a finite limit as $n \to \infty$.

For any $n \in \mathbb{N}$ let $a_n = (n^2+1)/n^2$ and $p_n = a_1 a_2 \dots a_n$. Then p_n is a partial product of the given infinite product. We have

$$\log p_n = \log(a_1 a_2 \dots a_n) = \log a_1 + \log a_2 + \dots + \log a_n.$$

Using inequality $\log x \le x - 1$, which holds for all $x > 0$, we obtain that $\log a_n \le a_n - 1 = 1/n^2$. Besides, $\log a_n > 0$ since $a_n > 1$. By the Comparison Test, the series $\sum_{n=1}^{\infty} \log a_n$ converges. Since $\log p_n$ is a partial sum of order n of this series, the sequence $\{\log p_n\}$ converges to a finite limit L. Then $p_n = e^{\log p_n} \to e^L$ as $n \to \infty$.