MATH 409

Advanced Calculus I

Lecture 4:
Intervals.
Principle of mathematical induction.
Inverse function.

Problem. Construct a strict linear order $\prec$ on the set $\mathbb{C}$ of complex numbers such that $a \prec b$ implies $a + c \prec b + c$ for all $a, b, c \in \mathbb{C}$.

Solution. Given complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ (where $x_1, x_2, y_1, y_2 \in \mathbb{R}$ and $i = \sqrt{-1}$), we let $z_1 \prec z_2$ if one of the following conditions hold:

- $x_1 < x_2$,
- $x_1 = x_2$ and $y_1 < y_2$.

It is easy to see that $\prec$ is a strict linear order on $\mathbb{C}$. Also, it is easy to check that $z_1 \prec z_2$ if and only if $0 \prec z_2 - z_1$. Hence $z_1 \prec z_2$ if and only if $z_1 + w \prec z_2 + w$ for all $z_1, z_2, w \in \mathbb{C}$.

Remark. The order $\prec$ is essentially an order on $\mathbb{R}^2$. An analogous order can be introduced on the set $\mathbb{R}^n$ for any $n > 1$. Such an order is called **lexicographic**, which refers to the ordering of words in a dictionary.

Problem. Construct a strict linear order $\prec$ on the set $\mathbb{R}(x)$ of rational functions in variable x with real coefficients that makes $\mathbb{R}(x)$ into an ordered field.

Solution. Given rational functions $f, g \in \mathbb{R}(x)$, we let $f \prec g$ if there exists $M \in \mathbb{R}$ such that $f(x) < g(x)$ for all $x > M$.

It is easy to observe that $\prec$ is a strict order on $\mathbb{R}(x)$. Also, it is easy to verify those axioms of an ordered field that involve addition and multiplication. The only hard part is to show that $\prec$ is a linear order.

Assume that $f \neq g$ and let $h = g - f$. Then $h(x) = p(x)/q(x)$, where p and q are nonzero polynomials. Since any nonzero polynomial has only finitely many roots, there exists $M \in \mathbb{R}$ such that p and q have no roots in the interval (M, ∞). The function h is continuous and nowhere zero on (M, ∞). Therefore it maintains its sign on this interval, that is, either $h(x) > 0$ for all $x > M$ or $h(x) < 0$ for all $x > M$. In the first case, $f \prec g$. In the second case, $g \prec f$.

General intervals

Suppose X is a set endowed with a strict linear order $\prec$. A subset $E \subset X$ is called an **interval** if with any two elements it contains all elements of X that lie between them. To be precise, $a, b \in E$ and $a \prec c \prec b$ imply that $c \in E$ for all $a, b, c \in X$.

Examples of intervals.

- The empty set and all one-element subsets of X are trivially intervals.
- Open finite interval $(a, b) = \{c \in X \mid a \prec c \prec b\}$, where $a, b \in X$, $a \prec b$.
- Closed and semi-open finite intervals $[a, b] = (a, b) \cup \{a, b\}$, $[a, b) = (a, b) \cup \{a\}$, and $(a, b] = (a, b) \cup \{b\}$.

General intervals

Suppose X is a set endowed with a strict linear order $\prec$. A subset $E \subset X$ is called an **interval** if with any two elements it contains all elements of X that lie between them. To be precise, $a, b \in E$ and $a \prec c \prec b$ imply that $c \in E$ for all $a, b, c \in X$.

Examples of intervals.

- Open semi-infinite intervals $(a, \infty) = \{c \in X \mid a \prec c\}$ and $(-\infty, a) = \{c \in X \mid c \prec a\}$, where $a \in X$.
- Closed semi-infinite intervals $[a, \infty) = (a, \infty) \cup \{a\}$ and $(-\infty, a] = (-\infty, a) \cup \{a\}$.
- The entire set X is an interval denoted $(-\infty, \infty)$.

In general, there might exist other types of intervals.

Intervals of the real line

Theorem 1 Suppose E is a bounded interval of $\mathbb{R}$ that consists of more than one point. Then there exist $a, b \in \mathbb{R}$, $a < b$, such that $E = (a, b)$ or $[a, b)$ or $(a, b]$ or $[a, b]$.

Theorem 2 Suppose E is an interval of $\mathbb{R}$ bounded above but unbounded below. Then there exists $a \in \mathbb{R}$ such that $E = (-\infty, a)$ or $(-\infty, a]$.

Theorem 3 Suppose E is an interval of $\mathbb{R}$ bounded below but unbounded above. Then there exists $a \in \mathbb{R}$ such that $E = (a, \infty)$ or $[a, \infty)$.

Theorem 4 Suppose E is an interval of $\mathbb{R}$ that is neither bounded above nor bounded below. Then $E = \mathbb{R}$.

Natural, integer, and rational numbers

Definition. A set $E \subset \mathbb{R}$ is called **inductive** if $1 \in E$ and, for any real number x, $x \in E$ implies $x + 1 \in E$. The set $\mathbb{N}$ of **natural numbers** is the smallest inductive subset of $\mathbb{R}$.

Remark. The set $\mathbb{N}$ is well defined. Namely, it is the intersection of all inductive subsets of $\mathbb{R}$.

The set of **integers** is defined as

$$\mathbb{Z} = -\mathbb{N} \cup \{0\} \cup \mathbb{N}.$$

The set of **rationals** is defined as

$$\mathbb{Q} = \{m/n \mid m \in \mathbb{Z}, n \in \mathbb{N}\}.$$

Basic properties of the natural numbers

- 1 is the least natural number.

The interval $[1, \infty)$ is an inductive set. Hence $\mathbb{N} \subset [1, \infty)$.

- If $n \in \mathbb{N}$, then $n - 1 \in \mathbb{N} \cup \{0\}$.

Let E be the set of all $n \in \mathbb{N}$ such that $n - 1 \in \mathbb{N} \cup \{0\}$. Then $1 \in E$ as $1 - 1 = 0$. Besides, for any $n \in E$ we have $(n + 1) - 1 = n \in \mathbb{N}$ so that $n + 1 \in E$. Therefore E is an inductive set. Then $\mathbb{N} \subset E$, which implies that $E = \mathbb{N}$.

- If $n \in \mathbb{N}$, then the open interval $(n - 1, n)$ contains no natural numbers.

Let E be the set of all $n \in \mathbb{N}$ such that $(n - 1, n) \cap \mathbb{N} = \emptyset$. Then $1 \in E$ as $\mathbb{N} \subset [1, \infty)$. Now assume $n \in E$ and take any $x \in (n, n + 1)$. We have $x - 1 \neq 0$ since $x > n \geq 1$, and $x - 1 \notin \mathbb{N}$ since $x - 1 \in (n - 1, n)$. By the above, $x \notin \mathbb{N}$. Thus E is an inductive set, which implies that $E = \mathbb{N}$.

Principle of well-ordering

Definition. Suppose X is a set endowed with a strict linear order $\prec$. We say that a subset $Y \subset X$ is **well-ordered** with respect to $\prec$ if any nonempty subset of Y has a least element.

Theorem The set $\mathbb{N}$ is well-ordered with respect to the natural ordering of the real line $\mathbb{R}$.

Proof: Let E be an arbitrary nonempty subset of $\mathbb{N}$. The set E is bounded below since 1 is a lower bound of $\mathbb{N}$. Therefore $m = \inf E$ exists. Since m is a lower bound of E while $m + 1$ is not, we can find $n \in E$ such that $m \leq n < m + 1$. As shown before, the interval $(n - 1, n)$ is disjoint from $\mathbb{N}$. Then $(-\infty, n) = (-\infty, m) \cup (n - 1, n)$ is disjoint from E, which implies that n is a lower bound of E. Hence $n \leq \inf E = m$ so that $n = m = \inf E$. Thus n is the least element of E.

Principle of mathematical induction

Theorem Let $P(n)$ be an assertion depending on a natural variable n. Suppose that

- $P(1)$ holds,
- whenever $P(k)$ holds, so does $P(k+1)$.

Then $P(n)$ holds for all $n \in \mathbb{N}$.

Proof: Let E be the set of all natural numbers n such that $P(n)$ holds. Clearly, E is an inductive set. Therefore $\mathbb{N} \subset E$, which implies that $E = \mathbb{N}$. ■

Remark. The assertion $P(1)$ is called the **basis of induction**. The implication $P(k) \implies P(k+1)$ is called the **induction step**.

Examples of assertions $P(n)$:

(a) $1 + 2 + \cdots + n = n(n+1)/2$,
(b) $n(n+1)(n+2)$ is divisible by 6,
(c) $n = 2p + 3q$ for some $p, q \in \mathbb{Z}$.

Theorem $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$.

Proof: The proof is by induction on n.

Basis of induction: check the formula for $n = 1$.
In this case, $1 = 1(1+1)/2$, which is true.

Induction step: assume that the formula is true for $n = m$ and derive it for $n = m + 1$.
Inductive assumption: $1 + 2 + \cdots + m = m(m+1)/2$.
Then

$$1 + 2 + \cdots + m + (m+1) = \frac{m(m+1)}{2} + (m+1)$$

$$= (m+1)\Big(\frac{m}{2} + 1\Big) = \frac{(m+1)(m+2)}{2}.$$

Strong induction

Theorem Let $P(n)$ be an assertion depending on a natural variable n. Suppose that $P(n)$ holds whenever $P(k)$ holds for all natural $k < n$. Then $P(n)$ holds for all $n \in \mathbb{N}$.

Remark. For $n = 1$, the assumption of the theorem means that $P(1)$ holds unconditionally. For $n = 2$, it means that $P(1)$ implies $P(2)$. For $n = 3$, it means that $P(1)$ and $P(2)$ imply $P(3)$. And so on...

Proof of the theorem: For any natural number n we define new assertion $Q(n) =$"$P(k)$ holds for any natural $k \leq n$". Then $Q(1)$ is equivalent to $P(1)$, in particular, it holds. By assumption, $Q(n)$ implies $P(n+1)$ for any $n \in \mathbb{N}$. Moreover, $Q(n+1)$ holds if and only if both $Q(n)$ and $P(n+1)$ hold. Therefore, $Q(n)$ implies $Q(n+1)$ for all $n \in \mathbb{N}$. By the principle of mathematical induction, $Q(n)$ holds for all $n \in \mathbb{N}$. Then $P(n)$ holds for all $n \in \mathbb{N}$ as well.

Functions

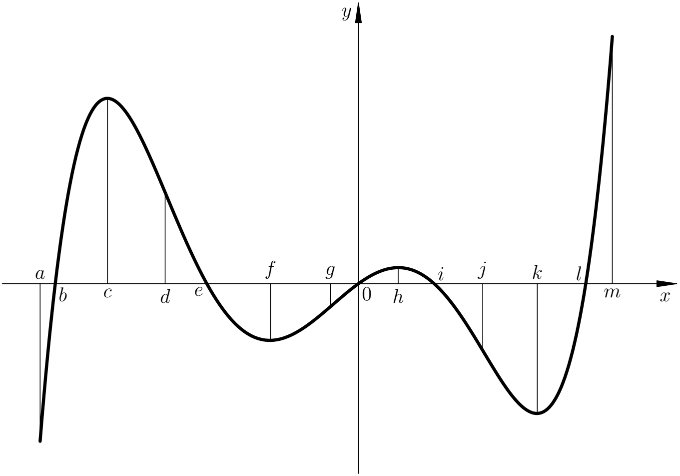
A **function** $f : X \to Y$ is an assignment: to each $x \in X$ we assign an element $f(x) \in Y$.

The **graph** of the function $f : X \to Y$ is defined as the subset of $X \times Y$ consisting of all pairs of the form $(x, f(x))$, $x \in X$. Two functions are considered the same if their graphs coincide.

Definition. A function $f : X \to Y$ is **surjective** (or **onto**) if for each $y \in Y$ there exists at least one $x \in X$ such that $f(x) = y$.

The function f is **injective** (or **one-to-one**) if $f(x') = f(x) \implies x' = x$.

Finally, f is **bijective** if it is both surjective and injective. Equivalently, if for each $y \in Y$ there is exactly one $x \in X$ such that $f(x) = y$.



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Inverse function

Suppose we have two functions $f : X \to Y$ and $g : Y \to X$. We say that g is the **inverse function** of f (denoted f^{-1}) if $y = f(x) \iff g(y) = x$ for all $x \in X$ and $y \in Y$.

Theorem 1 The inverse function f^{-1} exists if and only if f is bijective.

Theorem 2 A function $g : Y \to X$ is an inverse function of a function $f : X \to Y$ if and only if $g(f(x)) = x$ for all $x \in X$ and $f(g(y)) = y$ for all $y \in Y$.