MATH 409

Advanced Calculus I

Lecture 6: Limits of sequences. Limit theorems.

Convergence of a sequence

A **sequence** of elements of a set X is a function $f : \mathbb{N} \to X$. Notation: $x_1, x_2, \dots,$ where $x_n = f(n)$, or $\{x_n\}_{n\in\mathbb{N}}$, or $\{x_n\}$.

Definition. Sequence $\{x_n\}$ of real numbers is said to **converge** to a real number a if for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_n - a| < \varepsilon$ for all $n \geq N$. The number a is called the **limit** of $\{x_n\}$. Notation: $\lim\limits_{n\to\infty} x_n = a$, or $x_n \to a$ as $n \to \infty$.

A sequence is called **convergent** if it has a limit and **divergent** otherwise.

The condition $|x_n - a| < \varepsilon$ is equivalent to $a - \varepsilon < x_n < a + \varepsilon$ or to $x_n \in (a - \varepsilon, a + \varepsilon)$. The interval $(a - \varepsilon, a + \varepsilon)$ is called the ε-**neighborhood** of the point a. The convergence $x_n \to a$ means that any ε-neighborhood of a contains all but finitely many elements of the sequence $\{x_n\}$.

Examples

- The sequence $\{1/n\}_{n\in\mathbb{N}}$ converges to 0.

By the Archimedean Principle, for any $\varepsilon > 0$ there exists a natural number N such that $N\varepsilon > 1$. Then for any integer $n \geq N$ we have $n\varepsilon \geq N\varepsilon > 1$ so that $1/n < \varepsilon$. Since $1/n > 0$, we obtain $|1/n| < \varepsilon$ for all $n \geq N$.

- Constant sequence $\{x_n\}$, where $x_n = a$ for some $a \in \mathbb{R}$ and all $n \in \mathbb{N}$.

Since $|x_n - a| = 0$ for all $n \in \mathbb{N}$, the sequence converges to a.

- Sequence $\{(-1)^n\}_{n\in\mathbb{N}}$, i.e., $-1, 1, -1, 1, \dots,$ is divergent.

- Sequence $\{n\}_{n\in\mathbb{N}}$, i.e., $1, 2, 3, 4, \dots,$ is divergent.

Basic properties of convergent sequences

- The limit is unique.

Suppose a and b are distinct real numbers. Let $\varepsilon = |b - a|/2$. Then ε-neighborhoods of a and b are disjoint. Hence they cannot both contain all but finitely many elements of the same sequence.

- Any convergent sequence $\{x_n\}$ is **bounded**,

which means that the set of its elements is bounded.

This follows from three facts: any ε-neighborhood is bounded, any finite set is bounded, and the union of two bounded sets is also bounded.

- Any **subsequence** converges to the same limit.

Here a subsequence of a sequence $\{x_n\}_{n\in\mathbb{N}}$ is any sequence of the form $\{x_{n_k}\}_{k\in\mathbb{N}}$, where $\{n_k\}$ is an increasing sequence of natural numbers (note that $n_k \geq k$).

Divergence to infinity

Definition. A sequence $\{x_n\}$ is said to **diverge to infinity** if for any $C > 0$ there exists $N \in \mathbb{N}$ such that $|x_n| > C$ for all $n \geq N$.

Observe that such a sequence is indeed divergent as it is not bounded.

Definition. A sequence $\{x_n\}$ is said to diverge to $+\infty$ if for any $C \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that $x_n > C$ for all $n \geq N$. Likewise, $\{x_n\}$ is said to diverge to $-\infty$ if for any $C \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that $x_n < C$ for all $n \geq N$.

Example. The sequence $\{n\}_{n \in \mathbb{N}}$ diverges to $+\infty$.

Squeeze Theorem

Theorem Suppose $\{x_n\}$, $\{y_n\}$, and $\{w_n\}$ are three sequences of real numbers such that

$$x_n \le w_n \le y_n \quad \text{for all sufficiently large } n.$$

If the sequences $\{x_n\}$ and $\{y_n\}$ both converge to the same limit a, then $\{w_n\}$ converges to a as well.

Proof: Since $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = a$, for any $\varepsilon > 0$ there exist natural numbers N_1 and N_2 such that $a - \varepsilon < x_n < a + \varepsilon$ for all $n \ge N_1$ and $a - \varepsilon < y_n < a + \varepsilon$ for all $n \ge N_2$. Besides, there exists $N_0 \in \mathbb{N}$ such that $x_n \le w_n \le y_n$ for all $n \ge N_0$. Set $N = \max(N_0, N_1, N_2)$. Then for any natural number $n \ge N$ we have $a - \varepsilon < x_n \le w_n \le y_n < a + \varepsilon$, which implies that $a - \varepsilon < w_n < a + \varepsilon$. Thus $\lim_{n\to\infty} w_n = a$.

Comparison Theorem

Theorem Suppose $\{x_n\}$ and $\{y_n\}$ are convergent sequences. If $x_n \leq y_n$ for all sufficiently large n, then $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$.

Proof: Let $a = \lim_{n\to\infty} x_n$ and $b = \lim_{n\to\infty} y_n$. Assume to the contrary that $a > b$. Then $\varepsilon = (a-b)/2$ is a positive number. Hence there exists a natural number N such that $|x_n - a| < \varepsilon$ and $|y_n - b| < \varepsilon$ for all $n \geq N$. In particular, $y_n < b + \varepsilon$ and $a - \varepsilon < x_n$ for $n \geq N$. However $b + \varepsilon = a - \varepsilon = (a+b)/2$, which implies that $y_n < x_n$ for all $n \geq N$, a contradiction.

Corollary If all elements of a convergent sequence $\{x_n\}$ belong to a closed interval $[a, b]$, then the limit belongs to $[a, b]$ as well.

Convergence and arithmetic operations

Theorem Suppose $\{x_n\}$ and $\{y_n\}$ are convergent sequences of real numbers. Then the sequences $\{x_n + y_n\}_{n\in\mathbb{N}}$ and $\{x_n - y_n\}_{n\in\mathbb{N}}$ are also convergent.

Moreover, if $a = \lim_{n\to\infty} x_n$ and $b = \lim_{n\to\infty} y_n$, then

$$\lim_{n\to\infty}(x_n + y_n) = a + b \text{ and } \lim_{n\to\infty}(x_n - y_n) = a - b.$$

Proof: Since $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} y_n = b$, for any $\varepsilon > 0$ there exists a natural number N such that $|x_n - a| < \varepsilon/2$ and $|y_n - b| < \varepsilon/2$ for all $n \geq N$. Then for any $n \geq N$ we obtain

$$\begin{aligned} |(x_n + y_n) - (a + b)| &= |(x_n - a) + (y_n - b)| \\ &\leq |x_n - a| + |y_n - b| < \varepsilon/2 + \varepsilon/2 = \varepsilon, \\ |(x_n - y_n) - (a - b)| &= |(x_n - a) + (b - y_n)| \\ &\leq |x_n - a| + |b - y_n| = |x_n - a| + |y_n - b| < \varepsilon. \end{aligned}$$

Thus $x_n + y_n \to a + b$ and $x_n - y_n \to a - b$ as $n \to \infty$.

Theorem Suppose $\{x_n\}$ and $\{y_n\}$ are convergent sequences of real numbers. Then the sequence $\{x_n y_n\}_{n\in\mathbb{N}}$ is also convergent. Moreover, if $a = \lim_{n\to\infty} x_n$ and $b = \lim_{n\to\infty} y_n$, then $\lim_{n\to\infty} x_n y_n = ab$.

Proof: Since $x_n \to a$ and $y_n \to b$ as $n \to \infty$, for any $\delta > 0$ there exists $N(\delta) \in \mathbb{N}$ such that $|x_n - a| < \delta$ and $|y_n - b| < \delta$ for all $n \geq N(\delta)$. Then for any $n \geq N(\delta)$ we obtain

$$\begin{aligned}
|x_n y_n - ab| &= |x_n y_n - a y_n + a y_n - ab| = |(x_n - a) y_n + a(y_n - b)| \\
&= |(x_n - a) y_n - (x_n - a) b + (x_n - a) b + a(y_n - b)| \\
&= |(x_n - a)(y_n - b) + (x_n - a) b + a(y_n - b)| \\
&\leq |(x_n - a)(y_n - b)| + |(x_n - a) b| + |a(y_n - b)| \\
&= |x_n - a|\,|y_n - b| + |b|\,|x_n - a| + |a|\,|y_n - b| \\
&< \delta^2 + (|a| + |b|)\delta.
\end{aligned}$$

Now, given $\varepsilon > 0$, we set $\delta = \min(1, (1 + |a| + |b|)^{-1}\varepsilon)$. Then $\delta > 0$ and $\delta^2 + (|a| + |b|)\delta \leq (1 + |a| + |b|)\delta \leq \varepsilon$. By the above, $|x_n y_n - ab| < \varepsilon$ for all $n \geq N(\delta)$.

Theorem Suppose that a sequence $\{x_n\}$ converges to some $a \in \mathbb{R}$. If $a \neq 0$ and $x_n \neq 0$ for all $n \in \mathbb{N}$, then the sequence $\{x_n^{-1}\}_{n\in\mathbb{N}}$ converges to a^{-1}.

Proof: Since $x_n \to a$ as $n \to \infty$, for any $\delta > 0$ there exists $N(\delta) \in \mathbb{N}$ such that $|x_n - a| < \delta$ for all $n \geq N(\delta)$.

Given $\varepsilon > 0$, we set $\delta = \min(|a|/2, |a|^2\varepsilon/2)$. Then for any $n \geq N(\delta)$ we have $|x_n - a| < |a|/2$. Since

$$|a| \leq |a - x_n| + |x_n| = |x_n - a| + |x_n|,$$

it follows that $|x_n| \geq |a| - |x_n - a| > |a| - |a|/2 = |a|/2$.

Furthermore, for any $n \geq N(\delta)$ we obtain

$$\left|\frac{1}{x_n} - \frac{1}{a}\right| = \left|\frac{a - x_n}{ax_n}\right| = \frac{|x_n - a|}{|a|\,|x_n|} \leq \frac{2|x_n - a|}{|a|^2} < \frac{2\delta}{|a|^2} \leq \varepsilon.$$

We conclude that $1/x_n \to 1/a$ as $n \to \infty$.

Corollary 1 If $\lim_{n\to\infty} x_n = a$, then $\lim_{n\to\infty} cx_n = ca$ for any $c \in \mathbb{R}$.

Corollary 2 If $\lim_{n\to\infty} x_n = a$, then $\lim_{n\to\infty} (-x_n) = -a$.

Corollary 3 If $\lim_{n\to\infty} x_n = a$, $\lim_{n\to\infty} y_n = b$, and, moreover, $b \neq 0$ and $y_n \neq 0$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} x_n/y_n = a/b$.

Proof: Since $b \neq 0$ and $y_n \neq 0$ for all $n \in \mathbb{N}$, it follows that $y_n^{-1} \to b^{-1}$ as $n \to \infty$. Since $x_n/y_n = x_n y_n^{-1}$ for all $n \in \mathbb{N}$, we obtain that $x_n/y_n \to ab^{-1} = a/b$ as $n \to \infty$.