MATH 409

Advanced Calculus I

Lecture 9: Limit supremum and infimum. Limits of functions.

Limit points

Definition. A **limit point** of a sequence $\{x_n\}$ is the limit of any convergent subsequence of $\{x_n\}$.

Properties of limit points.

- A convergent sequence has only one limit point, its limit.
- Any bounded sequence has at least one limit point.
- If a bounded sequence is not convergent, then it has at least two limit points.
- If all elements of a sequence belong to a closed interval $[a, b]$, then all its limit points belong to $[a, b]$ as well.
- If a sequence diverges to infinity, then it has no limit points.
- If a sequence does not diverge to infinity, then it has a bounded subsequence and hence it has a limit point.

Limit supremum and infimum

Let $\{x_n\}$ be a bounded sequence of real numbers. For any $n \in \mathbb{N}$ let E_n denote the set of all numbers of the form x_k, where $k \geq n$. The set E_n is bounded, hence $\sup E_n$ and $\inf E_n$ exist. Observe that the sequence $\{\sup E_n\}$ is decreasing, the sequence $\{\inf E_n\}$ is increasing (since $E_1, E_2, \ldots$ are nested sets), and both are bounded. Therefore both sequences are convergent.

Definition. The limit of $\{\sup E_n\}$ is called the **limit supremum** of the sequence $\{x_n\}$ and denoted $\limsup\limits_{n\to\infty} x_n$.

The limit of $\{\inf E_n\}$ is called the **limit infimum** of the sequence $\{x_n\}$ and denoted $\liminf\limits_{n\to\infty} x_n$.

Properties of $\limsup$ *and* $\liminf$.

- $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$.

- $\liminf_{n\to\infty} x_n$ and $\limsup_{n\to\infty} x_n$ are limit points of the sequence $\{x_n\}$.

- All limit points of $\{x_n\}$ are contained in the interval $\left[\liminf_{n\to\infty} x_n, \limsup_{n\to\infty} x_n\right]$.

- The sequence $\{x_n\}$ converges to a limit a if and only if $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = a$.

Limit of a function

Let $I \subset \mathbb{R}$ be an open interval and $a \in I$. Suppose $f : E \to \mathbb{R}$ is a function defined on a set $E \supset I \setminus \{a\}$.

Definition. We say that the function f **converges to a limit** $L \in \mathbb{R}$ at the point a if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$0 < |x - a| < \delta \ \text{ implies } \ |f(x) - L| < \varepsilon.$$

Notation: $L = \lim_{x \to a} f(x)$ or $f(x) \to L$ as $x \to a$.

Remark. The set $(a - \delta, a) \cup (a, a + \delta)$ is called the **punctured δ-neighborhood** of a. Convergence to L means that, given $\varepsilon > 0$, the image of this set under the map f is contained in the ε-neighborhood $(L - \varepsilon, L + \varepsilon)$ of L provided that δ is small enough.

Limits of functions vs. limits of sequences

Theorem Let I be an open interval containing a point $a \in \mathbb{R}$ and f be a function defined on $I \setminus \{a\}$. Then $f(x) \to L$ as $x \to a$ if and only if for any sequence $\{x_n\}$ of elements of $I \setminus \{a\}$,

$$\lim_{n\to\infty} x_n = a \quad \text{implies} \quad \lim_{n\to\infty} f(x_n) = L.$$

Remark. Using this sequential characterization of limits, we can derive limit theorems for convergence of functions from analogous theorems dealing with convergence of sequences.

Limits of functions vs. limits of sequences

Proof of the theorem: Suppose that $f(x) \to L$ as $x \to a$. Consider an arbitrary sequence $\{x_n\}$ of elements of the set $I \setminus \{a\}$ converging to a. For any $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - L| < \varepsilon$ for all $x \in \mathbb{R}$. Further, there exists $N \in \mathbb{N}$ such that $|x_n - a| < \delta$ for all $n \geq N$. Then $|f(x_n) - L| < \varepsilon$ for all $n \geq N$. We conclude that $f(x_n) \to L$ as $n \to \infty$.

Conversely, suppose that $f(x) \not\to L$ as $x \to a$. Then there exists $\varepsilon > 0$ such that for any $\delta > 0$ the image of the punctured neighborhood $(a - \delta, a) \cup (a, a + \delta)$ of the point a under the map f is not contained in $(L - \varepsilon, L + \varepsilon)$. In particular, for any $n \in \mathbb{N}$ there exists a point $x_n \in (a - 1/n, a) \cup (a, a + 1/n)$ such that $x_n \in I$ and $|f(x_n) - L| \geq \varepsilon$. We have that the sequence $\{x_n\}$ converges to a and $x_n \in I \setminus \{a\}$. However $f(x_n) \not\to L$ as $n \to \infty$.

Limit theorems

Squeeze Theorem If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = L$ and $f(x) \le h(x) \le g(x)$ for all x in a punctured neighborhood of the point a, then $\lim_{x \to a} h(x) = L$.

Comparison Theorem If $\lim_{x \to a} f(x) = L$, $\lim_{x \to a} g(x) = M$, and $f(x) \le g(x)$ for all x in a punctured neighborhood of the point a, then $L \le M$.

Limit theorems

Theorem If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then

$$\lim_{x \to a} (f + g)(x) = L + M,$$
$$\lim_{x \to a} (f - g)(x) = L - M,$$
$$\lim_{x \to a} (fg)(x) = LM.$$

If, additionally, $M \neq 0$ then

$$\lim_{x \to a} (f/g)(x) = L/M.$$

Divergence to infinity

Let $I \subset \mathbb{R}$ be an open interval and $a \in I$. Suppose $f : E \to \mathbb{R}$ is a function defined on a set $E \supset I \setminus \{a\}$.

Definition. We say that the function f **diverges to** $+\infty$ at the point a if for every $C \in \mathbb{R}$ there exists $\delta = \delta(C) > 0$ such that

$$0 < |x - a| < \delta \text{ implies } f(x) > C.$$

Notation: $\lim_{x \to a} f(x) = +\infty$ or $f(x) \to +\infty$ as $x \to a$.

Similarly, we define the **divergence to** $-\infty$ at the point a.

One-sided limits

Let $f : E \to \mathbb{R}$ be a function defined on a set $E \subset \mathbb{R}$.

Definition. We say that f **converges to a right-hand limit** $L \in \mathbb{R}$ at a point $a \in \mathbb{R}$ if the domain E contains an interval (a, b) and for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$a < x < a + \delta \text{ implies } |f(x) - L| < \varepsilon.$$

Notation: $L = \lim_{x \to a+} f(x)$.

Similarly, we define the **left-hand limit** $\lim_{x \to a-} f(x)$.

Theorem $f(x) \to L$ as $x \to a$ if and only if $\lim_{x \to a+} f(x) = \lim_{x \to a-} f(x) = L$.

Limits at infinity

Let $f : E \to \mathbb{R}$ be a function defined on a set $E \subset \mathbb{R}$.

Definition. We say that f **converges to a limit** $L \in \mathbb{R}$ as $x \to +\infty$ if the domain E contains an interval $(a, +\infty)$ and for every $\varepsilon > 0$ there exists $C = C(\varepsilon) \in \mathbb{R}$ such that

$$x > C \quad \text{implies} \quad |f(x) - L| < \varepsilon.$$

Notation: $L = \lim_{x \to +\infty} f(x)$ or $f(x) \to L$ as $x \to +\infty$.

Similarly, we define the **limit** $\lim_{x \to -\infty} f(x)$.

Examples

- Constant function: $f(x) = c$ for all $x \in \mathbb{R}$ and some $c \in \mathbb{R}$.

$\lim_{x \to a} f(x) = c$ for all $a \in \mathbb{R}$. Also, $\lim_{x \to \pm\infty} f(x) = c$.

- Identity function: $f(x) = x$, $x \in \mathbb{R}$.

$\lim_{x \to a} f(x) = a$ for all $a \in \mathbb{R}$. Also, $\lim_{x \to +\infty} f(x) = +\infty$ and $\lim_{x \to -\infty} f(x) = -\infty$.

- Step function: $f(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$

$\lim_{x \to 0+} f(x) = 1$, $\lim_{x \to 0-} f(x) = 0$.

Examples

- $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}, \;\; f(x) = \dfrac{1}{x}.$

$\lim_{x\to a} f(x) = 1/a$ for all $a \neq 0$, $\lim_{x\to 0+} f(x) = +\infty$, $\lim_{x\to 0-} f(x) = -\infty$. Also, $\lim_{x\to \pm\infty} f(x) = 0$.

- $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}, \;\; f(x) = \sin \dfrac{1}{x}.$

$\lim_{x\to 0+} f(x)$ does not exist since $f((0,\delta)) = [-1,1]$ for any $\delta > 0$.

- $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}, \;\; f(x) = x \sin \dfrac{1}{x}.$

$\lim_{x\to 0} f(x) = 0$, which follows from the Squeeze Theorem since $-|x| \leq |f(x)| \leq |x|$.

Examples

- Dirichlet function: $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

$\lim_{x \to a} f(x)$ does not exist since $f((c,d)) = \{0,1\}$ for any interval (c,d). In other words, both rational and irrational points are dense in $\mathbb{R}$.

- Riemann function:

$$f(x) = \begin{cases} 1/q & \text{if } x = p/q, \text{ a reduced fraction,} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

$\lim_{x \to a} f(x) = 0$ for all $a \in \mathbb{R}$. Indeed, for any $n \in \mathbb{N}$ and a bounded interval (c,d), there are only finitely many points $x \in (c,d)$ such that $f(x) \geq 1/n$. On the other hand, $\lim_{x \to +\infty} f(x)$ and $\lim_{x \to -\infty} f(x)$ do not exist.