MATH 409

Advanced Calculus I

Lecture 11: More on continuous functions.

Continuity

Definition. Given a set $E \subset \mathbb{R}$, a function $f : E \to \mathbb{R}$, and a point $c \in E$, the function f is **continuous at** c if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $|x - c| < \delta$ and $x \in E$ imply $|f(x) - f(c)| < \varepsilon$.

We say that the function f is **continuous on** a set $E_0 \subset E$ if f is continuous at every point $c \in E_0$. The function f is **continuous** if it is continuous on the entire domain E.

Theorem A function $f : E \to \mathbb{R}$ is continuous at a point $c \in E$ if and only if for any sequence $\{x_n\}$ of elements of E, $x_n \to c$ as $n \to \infty$ implies $f(x_n) \to f(c)$ as $n \to \infty$.

Basic examples:

- Constant function: $f(x) = a$ for all $x \in \mathbb{R}$ and some $a \in \mathbb{R}$.
- Identity function: $f(x) = x$, $x \in \mathbb{R}$.

Theorem Suppose that functions $f, g : E \to \mathbb{R}$ are both continuous at a point $c \in E$. Then the functions $f + g$, $f - g$, and fg are also continuous at c. If, additionally, $g(c) \neq 0$, then the function f/g is continuous at c as well.

Examples of continuous functions:

- Power function: $f(x) = x^n$, $x \in \mathbb{R}$, where $n \in \mathbb{N}$.

Since the identity function is continuous and $x^{k+1} = x^k x$ for all $k \in \mathbb{N}$, it follows by induction on n that f is continuous.

- Polynomial: $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$.

Since constant functions and power functions are continuous, so are the functions $f_k(x) = a_k x^k$, $x \in \mathbb{R}$. Then f is continuous as a finite sum of continuous functions.

- Rational function: $f(x) = p(x)/q(x)$, where p and q are polynomials.

Since p and q are continuous, the function f is continuous on its entire domain $\{x \in \mathbb{R} \mid q(x) \neq 0\}$.

Extreme values and intermediate values

Theorem If $I = [a, b]$ is a closed, bounded interval of the real line, then any continuous function $f : I \to \mathbb{R}$ is bounded and attains its extreme values (maximum and minimum) on I.

Theorem If a function $f : [a, b] \to \mathbb{R}$ is continuous then any number y_0 that lies between $f(a)$ and $f(b)$ is a value of f, i.e., $y_0 = f(x_0)$ for some $x_0 \in [a, b]$.

Corollary If a real-valued function f is continuous on a closed bounded interval $[a, b]$, then the image $f([a, b])$ is also a closed bounded interval.

Theorem Any polynomial of odd degree has at least one real root.

Proof: Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of positive degree n. Note that $a_n \neq 0$. For any $x \neq 0$ we have

$$\frac{p(x)}{a_n x^n} = 1 + \frac{a_{n-1}}{a_n x} + \cdots + \frac{a_1}{a_n x^{n-1}} + \frac{a_0}{a_n x^n},$$

which converges to 1 as $x \to \pm\infty$. As a consequence, there exists $C > 0$ such that $p(x)/(a_n x^n) \geq 1/2$ if $|x| \geq C$. In particular, the numbers $p(x)$ and $a_n x^n$ are of the same sign if $|x| \geq C$. In the case n is odd, this implies that one of the numbers $p(C)$ and $p(-C)$ is positive while the other is negative. By the Intermediate Value Theorem, we have $p(x) = 0$ for some $x \in [-C, C]$.

Given a function $f : (a, b) \to \mathbb{R}$ and a point $c \in (a, b)$, let f_1 denote the restriction of f to the interval $(a, c]$ and f_2 denote the restriction of f to $[c, b)$.

Theorem The function f is continuous if and only if both restrictions f_1 and f_2 are continuous.

Proof: For any $x \in (a, c)$, the continuity of f at x is equivalent to the continuity of f_1 at x. Likewise, the continuity of f at a point $y \in (c, b)$ is equivalent to the continuity of f_2 at y. The function f is continuous at c if $f(x) \to f(c)$ as $x \to c$. The restriction f_1 is continuous at c if $f(x) \to f(c)$ as $x \to c-$. The restriction f_2 is continuous at c if $f(x) \to f(c)$ as $x \to c+$. Therefore f is continuous at c if and only if both f_1 and f_2 are continuous at c.

Example. The function $f(x) = |x|$ is continuous on $\mathbb{R}$.

Indeed, f concides with the function $g(x) = x$ on $[0, \infty)$ and with the function $h(x) = -x$ on $(-\infty, 0]$.

Continuity of the composition

Let $f : E_1 \to \mathbb{R}$ and $g : E_2 \to \mathbb{R}$ be two functions. If $f(E_1) \subset E_2$, then the composition $(g \circ f)(x) = g(f(x))$ is a well defined function on E_1.

Theorem If f is continuous at a point $c \in E_1$ and g is continuous at $f(c)$, then $g \circ f$ is continuous at c.

Proof: Let us use the sequential characterization of continuity. Consider an arbitrary sequence $\{x_n\} \subset E_1$ converging to c. We have to show that

$$(g \circ f)(x_n) \to (g \circ f)(c) \text{ as } n \to \infty.$$

Since the function f is continuous at c, we obtain that $f(x_n) \to f(c)$ as $n \to \infty$. Moreover, all elements of the sequence $\{f(x_n)\}$ belong to the set E_2. Since the function g is continuous at $f(c)$, we obtain that $g(f(x_n)) \to g(f(c))$ as $n \to \infty$.

Examples of continuous functions

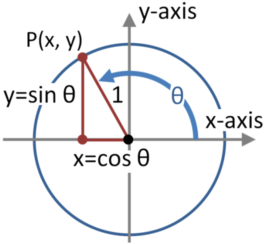
- If a function $f : E \to \mathbb{R}$ is continuous at a point $c \in E$, then a function $g(x) = |f(x)|$, $x \in E$, is also continuous at c.

Indeed, the function g is the composition of f with the continuous function $h(x) = |x|$.

- If functions $f, g : E \to \mathbb{R}$ are continuous at a point $c \in E$, then functions $\max(f, g)$ and $\min(f, g)$ are also continuous at c.

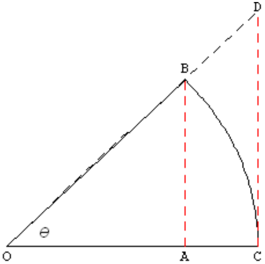
Indeed, $2\max(f(x), g(x)) = f(x) + g(x) + |f(x) - g(x)|$ and $2\min(f(x), g(x)) = f(x) + g(x) - |f(x) - g(x)|$ for all $x \in E$.

Trigonometric functions



$$\sin\theta = y$$
$$\cos\theta = x$$
$$\tan\theta = y/x$$

Theorem $0 \leq \sin\theta \leq \theta \leq \tan\theta$ for $\theta \in [0, \pi/2)$.



$$\sin\theta = |\text{segment } AB|$$
$$\theta = |\text{arc } CB|$$
$$\tan\theta = |\text{segment } CD|$$

Examples of continuous functions

- $f(x) = \sin x, \ x \in \mathbb{R}.$

We know that $0 \leq \sin\theta \leq \theta$ for $\theta \in [0, \pi/2)$. Since $\sin(-\theta) = -\sin\theta$, we get $|\sin\theta| \leq |\theta|$ if $|\theta| < \pi/2$. In the case $|\theta| \geq \pi/2$, this estimate holds too as $|\sin\theta| \leq 1 < \pi/2$. Now, using the trigonometric formula

$$\sin x - \sin c = 2 \sin \tfrac{x-c}{2} \cos \tfrac{x+c}{2},$$

we obtain $|\sin x - \sin c| \leq 2\,|\sin \tfrac{x-c}{2}|\,|\cos \tfrac{x+c}{2}| \leq 2\,|\tfrac{x-c}{2}|$ $= |x - c|$. It follows that $\sin x \to \sin c$ as $x \to c$ for every $c \in \mathbb{R}$. That is, the function $\sin x$ is continuous.

- $f(x) = \cos x, \ x \in \mathbb{R}.$

Since $\cos x = \sin(x + \pi/2)$ for all $x \in \mathbb{R}$, the function f is a composition of two continuous functions, $g(x) = x + \pi/2$ and $h(x) = \sin x$. Therefore it is continuous as well.

Examples of continuous functions

- $f(x) = \tan x$.

Since $f(x) = \dfrac{\sin x}{\cos x}$, the function f is continuous on its entire domain $\mathbb{R} \setminus \{x \in \mathbb{R} \mid \cos x = 0\} = \mathbb{R} \setminus \{\pi/2 + \pi k \mid k \in \mathbb{Z}\}$.

- $f(0) = 1$ and $f(x) = \dfrac{\sin x}{x}$ for $x \neq 0$.

Since $\sin x$ and the identity functions are continuous, it follows that f is continuous on $\mathbb{R} \setminus \{0\}$. Further, we know that $0 \leq \sin x \leq x \leq \tan x$ for $0 \leq x < \pi/2$. Therefore $\cos x \leq \dfrac{\sin x}{x} \leq 1$. Since $\cos 0 = 1$, the Squeeze Theorem implies that $f(x) \to 1$ as $x \to 0+$. The left-hand limit at 0 is the same as $f(-x) = f(x)$ for all $x \in \mathbb{R}$. Thus the function f is continuous at 0 as well.

Monotone functions

Let $f : E \to \mathbb{R}$ be a function defined on a set $E \subset \mathbb{R}$.

Definition. The function f is called **increasing** if, for any $x, y \in E$, $x < y$ implies $f(x) \leq f(y)$. It is called **strictly increasing** if $x < y$ implies $f(x) < f(y)$. Likewise, f is **decreasing** if $x < y$ implies $f(x) \geq f(y)$ and **strictly decreasing** $x < y$ implies $f(x) > f(y)$ for all $x, y \in E$.

Increasing and decreasing functions are called **monotone**. Strictly incresing and strictly decreasing functions are called **strictly monotone**.

Theorem 1 Any monotone function defined on an open interval can have only jump discontinuities.

Theorem 2 A monotone function f defined on an interval I is continuous if and only if the image $f(I)$ is also an interval.

Theorem 3 A continuous function defined on a closed interval is one-to-one if and only if it is strictly monotone.

Continuity of the inverse function

Suppose $f : E \to \mathbb{R}$ is a strictly monotone function defined on a set $E \subset \mathbb{R}$. Then f is one-to-one on E so that the **inverse function** f^{-1} is a well defined function on $f(E)$.

Theorem If the domain E of a strictly monotone function f is a closed interval and f is continuous on E, then the image $f(E)$ is also a closed interval, and the inverse function f^{-1} is strictly monotone and continuous on $f(E)$.

Proof: Since f is continuous on the closed interval E, it follows from the Extreme Value and Intermediate Value theorems that $f(E)$ is also a closed interval. The inverse function f^{-1} is strictly monotone since f is strictly monotone. By construction, f^{-1} maps the interval $f(E)$ onto the interval E, which implies that f^{-1} is continuous.

Examples

- Power function $f(x) = x^n$, $x \in \mathbb{R}$, where $n \in \mathbb{N}$.

The function f is continuous on $\mathbb{R}$. It is strictly increasing on the interval $[0, \infty)$ and $f([0, \infty)) = [0, \infty)$. In the case n is odd, the function f is strictly increasing on $\mathbb{R}$ and $f(\mathbb{R}) = \mathbb{R}$. We conclude that the inverse function $f^{-1}(x) = x^{1/n}$ is a continuous function on $[0, \infty)$ if n is even and a continuous function on $\mathbb{R}$ if n is odd.

- $f(x) = x^n$, $x \in \mathbb{R} \setminus \{0\}$, where $n \in \mathbb{Z}$, $n < 0$.

The function f is strictly decreasing on $(0, \infty)$. It is continuous on $(0, \infty)$ and maps this interval onto itself. Therefore the inverse function $f^{-1}(x) = x^{1/n}$ is a continuous function on $(0, \infty)$.