MATH 409

Advanced Calculus I

Lecture 17:
Applications of the mean value theorem.
l'Hôpital's rule.

Fermat's Theorem If a function f is differentiable at a point c of local extremum (maximum or minimum), then $f'(c) = 0$.

Rolle's Theorem If a function f is continuous on a closed interval $[a, b]$, differentiable on the open interval (a, b), and if $f(a) = f(b)$, then $f'(c) = 0$ for some $c \in (a, b)$.

Mean Value Theorem If a function f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that $f(b) - f(a) = f'(c)\,(b - a)$.

Theorem Suppose that a function f is continuous on $[a, b]$ and differentiable on (a, b). Then the following hold.

(i) f is increasing on $[a, b]$ if and only if $f' \geq 0$ on (a, b).
(ii) f is decreasing on $[a, b]$ if and only if $f' \leq 0$ on (a, b).
(iii) If $f' > 0$ on (a, b), then f is strictly increasing on $[a, b]$.
(iv) If $f' < 0$ on (a, b), then f is strictly decreasing on $[a, b]$.
(v) f is constant on $[a, b]$ if and only if $f' = 0$ on (a, b).

Examples

- $e^x > x + 1$ for all $x \neq 0$.

Consider a function $f(x) = e^x - x - 1$, $x \in \mathbb{R}$. This function is differentiable on $\mathbb{R}$ and $f'(x) = e^x - 1$ for all $x \in \mathbb{R}$. We observe that the derivative f' is strictly increasing. Since $f'(0) = 0$, we have $f'(x) < 0$ for $x < 0$ and $f'(x) > 0$ for $x > 0$. It follows that the function f is strictly decreasing on $(-\infty, 0]$ and strictly increasing on $[0, \infty)$. As a consequence, $f(x) > f(0) = 0$ for all $x \neq 0$. Thus $e^x > x + 1$ for $x \neq 0$.

- $\log x < x - 1$ for all $x > 0$, $x \neq 1$.

By the above, $e^{x-1} > (x - 1) + 1 = x$ for all $x \neq 1$. Since the natural logarithm is strictly increasing on $(0, \infty)$, it follows that $\log e^{x-1} > \log x$ for $x > 0$, $x \neq 1$. Equivalently, $\log x < x - 1$ for $x > 0$, $x \neq 1$.

Examples

- $(1-x)^\alpha > 1-\alpha x$ for all $x \in (0,1)$ and $\alpha > 1$.

Let us fix an arbitrary $\alpha > 1$ and consider a function

$$f(x) = (1-x)^\alpha - 1 + \alpha x, \ \ x \in [0,1).$$

This function is differentiable on $[0,1)$ and $f'(x) = -\alpha(1-x)^{\alpha-1} + \alpha$ for all $x \in [0,1)$. Since $\alpha - 1 > 0$, we obtain that $(1-x)^{\alpha-1} < 1$ for $x \in (0,1)$. Hence $f'(x) > 0$ for $x \in (0,1)$. It follows that the function f is strictly increasing on $[0,1)$. As a consequence, $f(x) > f(0) = 0$ for all $x \in (0,1)$. Equivalently, $(1-x)^\alpha > 1 - \alpha x$ for $x \in (0,1)$.

Examples

- $(1-x)^\alpha < 1 - \alpha x + \dfrac{\alpha(\alpha-1)}{2}x^2$ for all $x \in (0,1)$ and $\alpha > 2$.

Let us fix an arbitrary $\alpha > 2$ and consider a function

$$f(x) = (1-x)^\alpha - 1 + \alpha x - \tfrac{1}{2}\alpha(\alpha-1)x^2, \quad x \in [0,1).$$

This function is infinitely differentiable on $[0,1)$,
$f'(x) = -\alpha(1-x)^{\alpha-1} + \alpha - \alpha(\alpha-1)x$, and
$f''(x) = \alpha(\alpha-1)(1-x)^{\alpha-2} - \alpha(\alpha-1)$ for all $x \in [0,1)$.
Since $\alpha - 2 > 0$, we obtain that $f''(x) < 0$ for $x \in (0,1)$.
It follows that the derivative f' is strictly decreasing on $[0,1)$.
As a consequence, $f'(x) < f'(0) = 0$ for all $x \in (0,1)$.
Now it follows that the function f is also strictly decreasing on $[0,1)$. Consequently, $f(x) < f(0) = 0$ for all $x \in (0,1)$.
The required inequality follows.

Examples

- The function $f(x) = (1+x)^{1/x}$ is strictly decreasing on $(0, \infty)$.

Consider a function $g(x) = \log f(x)$, $x > 0$. For every $x > 0$, we have $g(x) = \log(1+x)/x$. Therefore g is differentiable on $(0, \infty)$ and $g'(x) = \left(\frac{x}{1+x} - \log(1+x)\right)/x^2$ for all $x > 0$. Now we introduce another function $h(x) = \frac{x}{1+x} - \log(1+x) = 1 - \frac{1}{1+x} - \log(1+x)$, $x \geq 0$. Note that $h(x) = x^2 g'(x)$ for $x > 0$. The function h is differentiable on $[0, \infty)$ and $h'(x) = \frac{1}{(1+x)^2} - \frac{1}{1+x} < 0$ for all $x > 0$. It follows that h is strictly decreasing on $[0, \infty)$. In particular, $h(x) < h(0) = 0$ for $x > 0$. Then $g'(x) < 0$ for $x > 0$ as well. Therefore g is strictly decreasing on $(0, \infty)$. Since the function f is the composition of g with the strictly increasing function $y(x) = e^x$, it is also strictly decreasing on $(0, \infty)$.

Taylor's formula

Theorem If a function $f : I \to \mathbb{R}$ is $n+1$ times differentiable on an open interval I, then for any two points $x, x_0 \in I$ there is a point c between x and x_0 such that

$$f(x) = f(x_0) + \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

Remark. The function

$$P_n^{f,x_0}(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

is a polynomial of degree at most n. It is called the **Taylor polynomial** of order n generated by f centered at x_0. Taylor's formula provides information on the remainder term $r_n^{f,x_0} = f - P_n^{f,x_0}$. In many cases this information allows to estimate $|r_n^{f,x_0}(x)|$ or to prove an inequality of the form $f(x) < P_n^{f,x_0}(x)$ or $f(x) > P_n^{f,x_0}(x)$.

l'Hôpital's Rule

l'Hôpital's Rule helps to compute limits of quotients in those cases where limit theorems do not apply (because of an indeterminacy of the form $0/0$ or ∞/∞).

Theorem Let a be either a real number or $-\infty$ or $+\infty$. Let I be an open interval such that either $a \in I$ or a is an endpoint of I. Suppose that functions f and g are differentiable on I and that $g(x), g'(x) \neq 0$ for $x \in I \setminus \{a\}$. Suppose further that

$$\lim_{\substack{x \to a \\ x \in I}} f(x) = \lim_{\substack{x \to a \\ x \in I}} g(x) = A,$$

where $A = 0$ or ∞. If the limit $\lim\limits_{\substack{x \to a \\ x \in I}} f'(x)/g'(x)$ exists (finite or infinite), then

$$\lim_{\substack{x \to a \\ x \in I}} \frac{f(x)}{g(x)} = \lim_{\substack{x \to a \\ x \in I}} \frac{f'(x)}{g'(x)}.$$

Remark. In fact, the theorem includes several similar rules corresponding to various kinds of limits ($\lim_{x\to a+}$, $\lim_{x\to a-}$, $\lim_{x\to a}$ for $a \in \mathbb{R}$, $\lim_{x\to+\infty}$, $\lim_{x\to-\infty}$) and the two types of indeterminacy ($0/0$ and ∞/∞).

Proof in the case $\lim_{x\to a+} 0/0$*:* We extend f and g to $I \cup \{a\}$ by letting $f(a) = g(a) = 0$. By hypothesis, f and g are continuous on $I \cup \{a\}$ and differentiable on I. By Generalized Mean Value Theorem, for any $x \in I$ there exists $c_x \in (a, x)$ such that

$$g'(c_x)\big(f(x) - f(a)\big) = f'(c_x)\big(g(x) - g(a)\big).$$

That is, $g'(c_x)f(x) = f'(c_x)g(x)$. Since $g(c_x), g'(c_x) \neq 0$, we obtain $f(x)/g(x) = f'(c_x)/g'(c_x)$. Since $c_x \in (a, x)$, we have $c_x \to a+$ as $x \to a+$. It follows that

$$\lim_{x\to a+} \frac{f(x)}{g(x)} = \lim_{x\to a+} \frac{f'(c_x)}{g'(c_x)} = \lim_{c\to a+} \frac{f'(c)}{g'(c)}.$$

Examples

- $\lim_{x \to 0} \dfrac{1 - \cos x}{x^2}$.

The functions $f(x) = 1 - \cos x$ and $g(x) = x^2$ are infinitely differentiable on $\mathbb{R}$. We have $\lim_{x \to 0} f(x) = f(0) = 0$ and $\lim_{x \to 0} g(x) = g(0) = 0$.

Further, $f'(x) = \sin x$ and $g'(x) = 2x$. We obtain $\lim_{x \to 0} f'(x) = f'(0) = 0$ and $\lim_{x \to 0} g'(x) = g'(0) = 0$.

Even further, $f''(x) = \cos x$ and $g''(x) = 2$. We obtain $\lim_{x \to 0} f''(x) = f''(0) = 1$ and $\lim_{x \to 0} g''(x) = g''(0) = 2$.
It follows that $\lim_{x \to 0} f''(x)/g''(x) = 1/2$.

By l'Hôpital's Rule, $\lim_{x \to 0} f'(x)/g'(x) = 1/2$. Applying l'Hôpital's Rule once again, we obtain $\lim_{x \to 0} f(x)/g(x) = 1/2$.

Examples

- $\lim_{x \to 0+} x^\alpha \log x$ and $\lim_{x \to +\infty} x^\alpha \log x$, where $\alpha \neq 0$.

We have $\lim_{x \to 0+} \log x = -\infty$ and $\lim_{x \to +\infty} \log x = +\infty$.

Besides, $\lim_{x \to 0+} x^{-\alpha} = 0$ if $\alpha < 0$ and $+\infty$ if $\alpha > 0$.

Since $1/x \to 0+$ as $x \to +\infty$, we obtain that
$\lim_{x \to +\infty} x^{-\alpha} = \lim_{x \to 0+} x^\alpha$.

It follows that $\lim_{x \to 0+} x^\alpha \log x = -\infty$ if $\alpha < 0$ and

$\lim_{x \to +\infty} x^\alpha \log x = +\infty$ if $\alpha > 0$.

Examples

- $\lim_{x \to 0+} x^{\alpha} \log x$ and $\lim_{x \to +\infty} x^{\alpha} \log x$, where $\alpha \neq 0$.

Further, we have $x^{\alpha} \log x = f(x)/g(x)$, where the functions $f(x) = \log x$ and $g(x) = x^{-\alpha}$ are infinitely differentiable on $(0, \infty)$. For any $x > 0$ we obtain $f'(x) = 1/x$ and $g'(x) = -\alpha x^{-\alpha-1}$. Hence $f'(x)/g'(x) = -\alpha^{-1} x^{\alpha}$ for all $x > 0$. Therefore in the case $\alpha < 0$ we have $\lim_{x \to 0+} f'(x)/g'(x) = +\infty$ and $\lim_{x \to +\infty} f'(x)/g'(x) = 0$.
In the case $\alpha > 0$, the two limits are interchanged.

By l'Hôpital's Rule, $\lim_{x \to 0+} f(x)/g(x) = 0$ if $\alpha > 0$ and $\lim_{x \to +\infty} f(x)/g(x) = 0$ if $\alpha < 0$.